

An Algebraic Approach to Form Factors

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Abstract

An associative $*$ -algebra is introduced (containing a $T\bar{T}R$ -algebra as a sub-algebra) that implements the form factor axioms, and hence indirectly the Wightman axioms, in the following sense: Each T -invariant linear functional over the algebra automatically satisfies all the form factor axioms. It is argued that this answers the question (posed in the functional Bethe ansatz) how to select the dynamically correct representations of the $T\bar{T}R$ -algebra. Applied to the case of integrable QFTs with diagonal factorized scattering theory a universal formula for the eigenvalues of the conserved charges emerges.

1. Introduction

There are two major approaches to construct and to solve integrable Quantum Field Theories (QFTs), the form factor bootstrap and the Quantum Inverse Scattering Method (QISM). Both methods are usually considered as being independent, with characteristic strengths and weaknesses. Let us briefly recapitulate the essentials of both techniques.

The form factor bootstrap [1, 2] takes an implementation of the Wightman axioms in terms of form factors as a starting point (for recent developments see [5, 6, 11, 12, 23, 7, 24] and below). Form factors are matrix elements of local operators between a multiparticle state and the physical vacuum. As a consequence of the factorized scattering theory there exists a recursive system of coupled Riemann-Hilbert equations for these form factors, which entail that the Wightman functions built from them have all the required properties. Given a factorized scattering theory the main problem in this approach is to identify the operator content of the model and to set up a correspondence to solutions of the form factor equations. This requires additional dynamical input. A distinguished infinite set of local operators are the conserved charges in involution. Their eigenvalues, once known, can thus serve to specify the additional dynamical input at least partially.

Usually the generating function for the eigenvalues of the local conserved charges in an integrable model is computed by means of the QISM. Principally the QISM achieves the construction of an integrable lattice model starting from a given classical integrable field theory (see [3, 4] and references therein). The dynamics is encoded into a representation of the celebrated $T\bar{T}R$ algebra and the QISM gives a prescription how to determine both, the algebra (i.e. the R matrix) and the representation class from the classical theory. In particular the representations relevant for QFTs are constructed as limits of finite dimensional representations (‘continuum limit of a lattice model’). On each of the finite dimensional representations the trace $t(\theta)$ of the monodromy operator can be diagonalized by Bethe Ansatz techniques. For the ‘correct’ R matrix and the ‘correct’ representation the eigenstates of $t(\theta)$ can then be interpreted as the asymptotic multi-particle states and the eigenvalues are generating functions for the eigenvalues of the conserved charges. From the viewpoint of relativistic QFTs the main shortcomings of the QISM are:

- It does not apply to models where the dynamically correct representation cannot be built from (algebraic) Bethe Ansatz techniques. Examples are the chiral sigma models and the real coupling affine Toda theories.
- It does not guarantee that a bona-fide relativistic QFT emerges that satisfies the Wightman axioms.

The purpose of this paper is to give an algebraic formulation of the form factor bootstrap that allows one to address the diagonalization problem of the conserved charges in the context of form factors. By the design of the form factors one expects the above technical shortcomings of the QISM to be absent. The main ingredient in this algebraic formulation is a doublet of ‘form factor’ algebras $F_{\pm}(S)$ associated with a given two particle bootstrap S -matrix. It has the following features:

1. It applies to any $1 + 1$ dimensional relativistic QFT with a mass gap and factorized scattering theory.
2. It contains an algebra of $TT\bar{R}$ -type as a subalgebra, where R is the physical S matrix.
3. It implements the form factor axioms (and hence indirectly the Wightman axioms) in the sense that any T -invariant linear functional f^{\pm} over $F_{\pm}(S)$ will automatically solve all the form factor axioms, except for the residue axioms. The sum $f = f^{+} + f^{-}$ then in addition satisfies the kinematical residue axiom.
4. Any T -invariant linear functional over $F_{\pm}(S)$ solves a system of linear difference equations (deformed Knizhnik-Zamolodchikov equations (KZE)).

Let us comment on these features. Certain fragments of such an algebra appeared in the context of Yangian and quantum double constructions studied by F. Smirnov, D. Bernard and A. LeClair [5, 6]. In particular Smirnov considered realizations of Yangians in terms of vertex operators such that suitable functionals over the algebra satisfied the deformed KZE and could asymptotically be set in correspondence to form factors[8, 9, 10]. Here we try to separate the algebraic and the representation theoretical aspects. It is remarkable that an algebra $F_{\pm}(S)$ exists that implements *all* the form factor axioms for *any* choice of representation (generically not of Fock-type) and for *any* factorized scattering theory. The only condition on the functionals over the algebra needed is

$$\begin{aligned} f(X T^{+}(\theta)_a^b) &= \delta_a^b f(X) \\ f(T^{-}(\theta)_a^b X) &= \omega(a) \delta_a^b f(X), \quad \theta \in \mathbb{C}, \end{aligned}$$

which we refer to as T -invariance. Here X is any element of $F_{\pm}(S)$, the generators being $W_a^{+}(\theta)$, $T^{\pm}(\theta)_a^b$ and $W_a^{-}(\theta)$, $T^{\pm}(\theta)_a^b$, respectively. In the second line $\omega(a)$ is a phase that depends only on f and a but not on X . For elements $X^{\pm} \in F_{\pm}(S)$ of the form $X^{\pm} = W_{a_n}^{\pm}(\theta_n) \dots W_{a_1}^{\pm}(\theta_1)$ write $f^{\pm}(X^{\pm}) = f_{a_n \dots a_1}^{\pm}(\theta_n, \dots, \theta_1)$. Given any doublet of T -invariant functionals f^{\pm} over $F_{\pm}(S)$, respectively, the claim in point three above is that

$$f_{a_n \dots a_1}(\theta_n, \dots, \theta_1) := f^{+}(X^{+}) + f^{-}(X^{-})$$

satisfies all the form factor axioms, except the bound state residue axiom. We expect that also the latter can be implemented algebraically, but since the complete set of bound state poles is strongly model dependent, this is best deferred to case studies. Feature three thus means that the solution of the infinite recursive system of form factor equations can be replaced by the study of the representation theory of the algebra $F_{\pm}(S)$. The task of investigating the representation theory of $F_{\pm}(S)$ should be facilitated by feature four, since one can exploit the existing body of knowledge on deformed KZE [19, 20, 21, 22]. The second feature, finally, means that one can address the diagonalization problem of the trace of the monodromy operator also in the context of form factors. By the design of the form factor doublet $F_{\pm}(S)$ the above technical shortcomings of the QISM should be overcome. In particular, there seems to be a simple answer to the question [13, 14] how to select the dynamically correct representations of the $T\bar{T}R$ (here $T\bar{T}S$) algebra: The dynamically correct representations are those that can be extended to T -invariant representations of $F_{\pm}(S)$. In order to test this criterion we applied it to compute the spectrum of the conserved charges in QFTs with diagonal factorized scattering theory. In the case of affine Toda theories we recover our previous result [15]. At least for QFTs with diagonal factorized scattering theory one thus has available a diagonalization method independent of, and alternative to, Bethe Ansatz techniques.

The paper is organized as follows. In section 2 we introduce the form factor doublet $F_{\pm}(S)$ and study some additional structures on it. The above features of the form factor algebra are derived in section 3. The criterion how to select the dynamically correct representation of the $T\bar{T}R$ -algebra and its application to QFTs with diagonal factorized scattering theory is discussed in section 4. The algebra $F_{\pm}(S)$ is defined for complex rapidities and bears no obvious relation to the Zamolodchikov-Faddeev (ZF) algebra, defined for real rapidities. In the appendix we study the relation between both algebras and prove the algebraic consistency of $F_{\pm}(S)$ and a number of related algebras.

2. The form factor algebra

2.1 Definition of the algebra

Let $S_{ab}^{dc}(\theta)$, $\theta \in \mathbb{C}$ be a physical two particle S -matrix i.e. a solution of the following set of equations. First, the Yang Baxter equation

$$S_{ab}^{nm}(\theta_{12})S_{nc}^{kp}(\theta_{13})S_{mp}^{ji}(\theta_{23}) = S_{bc}^{nm}(\theta_{23})S_{am}^{pi}(\theta_{13})S_{pn}^{kj}(\theta_{12}) , \quad (2.1)$$

where $\theta_{12} = \theta_1 - \theta_2$ etc. Second, unitarity (2.2a,b) and crossing invariance (2.2c)

$$S_{ab}^{mn}(\theta) S_{nm}^{cd}(-\theta) = \delta_a^d \delta_b^c \quad (2.2a)$$

$$S_{an}^{mc}(\theta) S_{bm}^{nd}(2\pi i - \theta) = \delta_a^d \delta_b^c \quad (2.2b)$$

$$S_{ab}^{dc}(\theta) = C_{aa'} C^{dd'} S_{bd'}^{ca'}(i\pi - \theta) , \quad (2.2c)$$

where (2.2c) together with (2.2a), (2.2b) implies (2.2b), (2.2a), respectively. Further, real analyticity and bose symmetry

$$[S_{ab}^{dc}(\theta)]^* = S_{ab}^{dc}(-\theta^*) , \quad (2.3)$$

$$S_{ab}^{dc}(\theta) = S_{ba}^{cd}(\theta) . \quad (2.4)$$

Finally, the normalization condition

$$S_{ab}^{dc}(0) = \epsilon_{ab} \delta_a^d \delta_b^c , \quad \epsilon_{ab} \in \{\pm 1\} , \quad \epsilon_{aa} = -1 . \quad (2.5)$$

It is convenient to borrow Penrose's abstract index notation from general relativity [16]. That is to say, indices a, b, \dots are not supposed to take numerical values but merely indicate the tensorial character of the quantity carrying it. Vectors v^a, v^b, \dots for example are elements of (classes of) abstract modules V^a, V^b, \dots of possibly different dimensionality. Covectors v_a, v_b, \dots are elements of the dual modules V_a, V_b, \dots and repeated upper and lower case indices indicate the duality pairing. Indices can be raised and lowered by means of the constant 'charge conjugation matrix' C_{ab} and its inverse C^{ab} , satisfying $C_{ad}C^{db} = \delta_a^b$.

To any solution of the Yang Baxter eqn. and the conditions (2.2), (2.5) consider the associative algebra generated by $(T^\pm)_a^b(\theta) =: T^\pm(\theta)_a^b$, $\theta \in \mathbb{C}$, and a unit $\mathbb{1}$ subject to the relations

$$\begin{aligned}
\text{(T1)} \quad & S_{ab}^{mn}(\theta_{12}) T^{\pm}(\theta_2)_n^d T^{\pm}(\theta_1)_m^c = T^{\pm}(\theta_1)_a^m T^{\pm}(\theta_2)_b^n S_{mn}^{cd}(\theta_{12}) , \\
& S_{ab}^{mn}(\theta_{12}) T^{\pm}(\theta_2)_n^d T^{\mp}(\theta_1)_m^c = T^{\mp}(\theta_1)_a^m T^{\pm}(\theta_2)_b^n S_{mn}^{cd}(\theta_{12}) ,
\end{aligned}$$

valid for all values of θ_{12} . Further

$$\begin{aligned}
\text{(T2)} \quad & C_{mn} T^{\pm}(\theta + i\pi)_a^m T^{\pm}(\theta)_b^n = C_{ab} \mathbb{1} , \\
& C^{mn} T^{\pm}(\theta)_m^a T^{\pm}(\theta + i\pi)_n^b = C^{ab} \mathbb{1} .
\end{aligned}$$

$T(S)$ can be given the structure of a Hopf algebra with antipode, comultiplication and counit given by*

$$\begin{aligned}
sT^{\pm}(\theta)_a^b &= C_{aa'} C^{bb'} T^{\pm}(\theta + i\pi)_{b'}^{a'} , \\
\Delta T^{\pm}(\theta)_a^b &= T^{\pm}(\theta)_a^m T^{\pm}(\theta)_m^b , \\
\epsilon T^{\pm}(\theta)_a^b &= 1 .
\end{aligned} \tag{2.6}$$

Now extend the algebra $T(S)$ by generators $W_a(\theta)$, $0 \leq \text{Im}\theta \leq 2\pi$ having the following linear exchange relations with $T^{\pm}(\theta)_a^b$

$$\text{(TW)} \quad T^{\pm}(\theta_0)_a^b W_{a_1}(\theta_1) = S_{aa_1}^{db_1}(\theta_{01}) W_{b_1}(\theta_1) T^{\pm}(\theta_0)_d^b .$$

We remark that no contractions are allowed in these relations (c.f. appendix A.3). Further impose

$$\text{(WW)} \quad W_a(\theta_1) W_b(\theta_2) = S_{ab}^{dc}(\theta_{12}) W_c(\theta_2) W_d(\theta_1) , \quad \text{Re } \theta_{12} \neq 0 .$$

The W -generators so far are defined only in the strip $0 \leq \text{Im}\theta \leq 2\pi$. The extension to other strips $2\pi k \leq \text{Im}\theta \leq 2\pi(k+1)$, $k \in \mathbb{Z}$ is done by repeated use of the relation

$$\text{(S)} \quad C^{mn} W_m(\theta) T^+(\theta + i\pi)_n^a = C^{mn} T^-(\theta + i\pi)_n^a W_m(\theta + 2\pi i) ,$$

which can be viewed as a deformed contracted version of (TW). Observe that (S) is equivalent to

$$\begin{aligned}
W_a(\theta + 2\pi i) &= T^-(\theta + 2\pi i)_a^m W_n(\theta) sT^+(\theta)_m^n , \\
W_a(\theta - 2\pi i) &= sT^-(\theta - 2\pi i)_m^n W_n(\theta) T^+(\theta - 2\pi i)_a^m .
\end{aligned} \tag{2.7}$$

*In particular s is a linear (not antilinear) anti-homomorphism. The relations (T1), (T2) for $T(S)$ coincide with that of a quantum double in its multiplicative presentation[18]. The relations (TW) are characteristic for intertwining operators between quantum double modules. We refrain from using this language because the relation (S) and its consequences do not seem to have a natural interpretation in the context of quantum doubles.

On the W generators the analytic continuation $\theta \rightarrow \theta + 2\pi i$ is thus implemented by an inner automorphism of the algebra. (Whereas only Lorentz boosts with real rapidities are unitarily implemented via the 1+1 dim. Poincaré group.) In summary, for any solution S of equations (2.1)–(2.5) we define an associative algebra $F_*(S)$ with generators $W_a(\theta)$, $(T^\pm)_a^b(\theta) =: T^\pm(\theta)_a^b$, $\theta \in \mathbb{C}$, a unit $\mathbb{1}$ and the generators P_μ , $\epsilon_{\mu\nu}K$ of the 1+1 dimensional Poincaré algebra. Except for P_μ all generators transform as scalars under the action of the Poincaré group. The defining relations of $F_*(S)$ then are that of $T(S)$ together with (TW), (WW) and (S).

The product of W generators in $F_*(S)$ is defined only when all relative rapidities have a nonvanishing real part. For relative rapidities that are purely imaginary, the product of W -generators contains simple poles. In particular $W_a(\theta_1)W_b(\theta_2)$ contains a simple pole at $\theta_{12} = \pm i\pi$. The algebra $F_*(S)$ in which the W -generators in addition satisfy the relation (R \pm) below will be denoted by $F_\pm(S)$, respectively. It is convenient to use different symbols $W_a^+(\theta)$ and $W_a^-(\theta)$ for W -generators satisfying (R+) and (R–), respectively. The residue conditions then read

$$\begin{aligned} \text{(R}\pm\text{)} \quad & 2\pi i \operatorname{res}[W_a^+(\theta - i\pi) W_b^+(\theta)] = -C_{ab} , \\ & 2\pi i \operatorname{res}[W_a^-(\theta + i\pi) W_b^-(\theta)] = -C_{ab} . \end{aligned}$$

We shall refer to the algebra $F_\pm(S)$ as the *form factor doublet*. Let us remark that multiple products of W -generators have been defined only in cases where at most one relative rapidity is purely imaginary. The extension of the product to cases where two or more relative rapidities are purely imaginary is tricky and will not be needed. Implicit in these definitions, of course, is the presupposition that the above relations define a consistent algebra. The verification of this fact is deferred to appendix A. The significance of $F_\pm(S)$ in the context of form factors has been outlined in the introduction and will be detailed in section 3.

2.2 The *-operation

In technical terms a *-operation is an antilinear anti-involution of some associative algebra. Here we shall denote such operations by σ since $*$ is already used for complex conjugation. The algebra $F_*(S)$ turns out to admit an antilinear anti-involution σ given by

$$\sigma(T^\pm)_a^b(\theta) = T^\mp(\theta^*)_a^b , \quad \sigma(W_a)(\theta) = W_a(\theta^*) , \quad \sigma^2 = id . \quad (2.8)$$

The same holds for the form factor algebra with $\sigma(W_a^\pm)(\theta) = W_a^\pm(\theta^*)$. The operations: ‘application of σ ’ and ‘taking the residue’ commute in (R \pm).

Usually in a $*$ -algebra of bounded operators any linear form f over the algebra can be used to define a sesquilinear form contravariant w.r.t. σ (the $*$ -operation of the algebra) by $(Y, X) := f(\sigma(Y)X)$. In the case at hand, the algebra $F_{\pm}(S)$ does not consist of bounded operators and the usual device to smear the operators with appropriate test functions is problematic, too. The reason is that we wish to keep track of the analyticity properties of expressions like $f(W_{a_1}(\theta_1) \dots W_{a_n}(\theta_n))$ as a function of $\theta_1, \dots, \theta_n$. In general such expressions will be germs of multivalued analytic functions with branch cuts and singularities, so that the construction of cycles (in the sense of integration theory) will be a non-trivial task. Rather than attempting to construct such integration cycles, it is technically much simpler to generalize the notion of a linear form instead. Thus we shall consider linear mappings

$$f : F_{\pm}(S) \longrightarrow G ,$$

where G is the space of germs of multivalued analytic functions in any number of complex variables. Alternatively one may think of these mappings as linear forms in the usual sense with the extra condition that the dependence on the rapidity variables parametrizing the elements of $F_{\pm}(S)$ is locally analytic. We shall refer to such maps as ‘analytic linear forms over $F_{\pm}(S)$ ’. Given any analytic linear form in that sense one can use the antilinear anti-involution σ to define the associated sesquilinear form contravariant w.r.t. it via

$$F(Y, X) := f(\sigma(Y) X) . \quad (2.9)$$

2.3 Residue equations

The relation (S) allows one to compute the residue of $W_a^{\pm}(\theta_1) W_b^{\pm}(\theta_2)$ for $\theta_{12} = \pm i\pi$, given that at $\theta_{12} = \mp i\pi$ in $(R\pm)$. Using (S) and (TW) one finds

$$\begin{aligned} 2\pi i \operatorname{res}[W_a^+(\theta + i\pi) W_b^+(\theta)] &= L_{ab}^+(\theta) , \\ 2\pi i \operatorname{res}[W_a^-(\theta - i\pi) W_b^-(\theta)] &= L_{ab}^-(\theta - i\pi) , \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} L_{ab}^+(\theta) &= C_{mn} T^-(\theta + i\pi)_a^m T^+(\theta)_b^n , \\ L_{ab}^-(\theta) &= C_{mn} T^+(\theta)_a^m T^-(\theta + i\pi)_b^n . \end{aligned} \quad (2.11)$$

For S -matrices where $S_{ab}^{dc}(\pm i\pi)$ is regular, these operators satisfy

$$L_{ab}^{\pm}(\theta) = -S_{ab}^{dc}(\pm i\pi) L_{cd}^{\mp}(\theta) , \quad (2.12)$$

using (T1) and $S_{na}^{bn}(2\pi i) = -\delta_a^b = S_{na}^{bn}(0)$. In particular, these equations imply that for purely imaginary relative rapidities the (WW) relations break down for $W_a^+(\theta)$ and $W_a^-(\theta)$.

The relations (R \pm) and (2.10) also allow one to make contact to a residue prescription first proposed by Smirnov[8]. Suppose that in addition to the previous relations one postulates

$$W_a^+(\theta_1) W_b^-(\theta_2) = S_{ab}^{dc}(\theta_{12}) W_c^-(\theta_2) W_d^+(\theta_1) , \quad Re \theta_{12} \neq 0 , \quad (2.13)$$

and

$$2\pi i \text{res}[W_a^\pm(\theta + i\pi) W_b^\mp(\theta)] = 0 . \quad (2.14)$$

The linear combination

$$W_a(\theta) := W_a^+(\theta) + W_a^-(\theta)$$

then will again satisfy the relations (TW), (S) and (WW). For the residues one finds from (R \pm) and (2.10), (2.14)

$$\begin{aligned} 2\pi i \text{res}[W_a(\theta + i\pi) W_b(\theta)] &= L_{ab}^+(\theta) - C_{ab} , \\ 2\pi i \text{res}[W_a(\theta) W_b(\theta + i\pi)] &= L_{ab}^-(\theta) - C_{ab} , \end{aligned} \quad (2.15)$$

so that by (2.12) a version of (WW) exchange relations is restored even (for the residues) at relative rapidities $\pm i\pi$. Operator-valued residue equations of the form (2.15) first appeared in [8]. If one postulates these relations at a fundamental level, however, they give little insight "how the W -generators manage to have such a residue". In particular, it would be difficult to construct realizations with this property. A compelling feature of the relation (S) is that, together with the simpler numerical residue equations (R \pm) and (R), they imply (2.15).

2.4 n -th roots of the $\theta \rightarrow \theta + n\pi i$ automorphism

The relation (S) also allows one to define a remarkable linear (not antilinear) anti-homomorphism \bar{s} on $F_*(S)$. Define

$$\bar{s}T^\pm(\theta)_a^b = C_{aa'} C^{bb'} T^\mp(\theta + i\pi)_{b'}^{a'} , \quad (2.16a)$$

$$\bar{s}W_a(\theta) = W_m(\theta) sT^+(\theta)_a^m = sT^-(\theta)_a^m W_m(\theta + 2\pi i) . \quad (2.16b)$$

Note that the restriction of \bar{s} to $T(S)$ is not the antipode map in (2.6), but differs from it by the interchange of $T^+(\theta)$ and $T^-(\theta)$. We claim that $\bar{s} : F_*(S) \rightarrow F_*(S)$ is a linear

anti-homomorphism that squares to the $\theta \rightarrow \theta + 2\pi i$ automorphism i.e.

$$\bar{s}^2 T^\pm(\theta)_a^b = T^\pm(\theta + 2\pi i)_a^b, \quad \bar{s}^2(W_a)(\theta) = W_a(\theta + 2\pi i). \quad (2.17)$$

Moreover, the \bar{s} -transformed W generators commute with the original ones

$$\bar{s}W_a(\theta_1)W_b(\theta_2) = W_b(\theta_2)\bar{s}W_a(\theta_1), \quad (2.18)$$

and may be viewed as an algebraic analogue of ‘screening operators’.

The equations (2.17), (2.18) follow directly from the definition. Observe that (2.18) can also be rewritten in the form

$$W_a(\theta_1 + 2\pi i)W_b(\theta_2) = S_{ab}^{dc}(\theta_{12} + 2\pi i) W_c(\theta_2)W_d(\theta_1 + 2\pi i), \quad (2.19)$$

where equation (2.7) is inserted for $W_a(\theta_1 + 2\pi i)$. The fact that \bar{s} is an anti-homomorphism is well-known for the $T(S)$ subalgebra; for the (TW) relations it amounts to

$$\bar{s}T^\pm(\theta_0)_a^b \bar{s}W_{a_1}(\theta_1) = S_{aa_1}^{db_1}(\theta_{10}) \bar{s}W_{b_1}(\theta_1) \bar{s}T^\pm(\theta_0)_d^b,$$

which one can verify for both expressions on the r.h.s. of (2.16b). For the (WW) relations there are correspondingly four cases to be checked. One finds consistently

$$\bar{s}W_a(\theta_1)\bar{s}W_b(\theta_2) = S_{ab}^{dc}(\theta_{21})\bar{s}W_c(\theta_2)\bar{s}W_d(\theta_1).$$

Finally, \bar{s} acts on (S) as an anti-homomorphism if $T^-(\theta)_a^m \bar{s}W_m(\theta - 2\pi i) = \bar{s}W_m(\theta)T^+(\theta)_a^m$ holds; which by (S) indeed is an identity. This shows that \bar{s} is an anti-homomorphism of $F_*(S)$.

It may be useful to compare \bar{s} to the adjoint action on a quantum double. Rewriting the usual definition in terms the generators $T^\pm(\theta)_a^b$ one obtains

$$Ad(T^\pm(\theta)_a^b) X := T^\pm(\theta)_a^d X sT^\pm(\theta - 2\pi i)_d^b, \quad (2.20)$$

where X is itself an element of the double. By means of (T2) the operation (2.20) has the characteristic properties of an adjoint action. In the matrix presentation employed here, the same formula can be used to define an adjoint action of $T(S)$ on the W -generators, which in the context of a double construction play the role of intertwining operators. The

defining relations for such an intertwiner (see e.g. [19, 6]) can be checked to be equivalent to our (TW) relations. For the above adoint action (TW) implies

$$\begin{aligned} Ad(T^\pm(\theta_0)_a^b)[W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1)] \\ = S_{aa_n}^{c_n b_n}(\theta_{0n}) S_{c_n a_{n-1}}^{c_{n-1} b_{n-1}}(\theta_{0n-1}) \dots S_{c_2 a_1}^{c_1 b_1}(\theta_{01}) W_{b_n}(\theta_n) \dots W_{b_1}(\theta_1) , \end{aligned}$$

valid for generic rapidities. Thus, besides being structurally different, the relation (S) concerns just those contractions where the (TW) relations break down (c.f. appendix A.3).

Consider now

$$sT^\pm(\theta)_a^b = C_{aa'} C^{bb'} T^\pm(\theta + i\pi)_{b'}^{a'} , \quad (2.21a)$$

$$sW_a(\theta) = W_m(\theta) sT^+(\theta)_a^m = sT^-(\theta)_a^m W_m(\theta + 2\pi i) . \quad (2.21b)$$

The restriction of s to $T(S)$ is the usual antipode map, while on the W -generators s and \bar{s} coincide $sW_a(\theta) = \bar{s}W_a(\theta)$. In particular, s again acts as an anti-homomorphism on $T(S)$ and the (TW) and (WW) relations. The condition that s acts as an anti-homomorphism on (S) reads

$$(S^2) \quad T^+(\theta + 2\pi i)_a^m sW_m(\theta) = sW_m(\theta + 2\pi i) T^-(\theta + 2\pi i)_a^m ,$$

which however is not an identity in $F_*(S)$. To cure this problem one may consider the algebra where (S^2) has been added to the defining relations. Then s is an anti-homomorphism of this modified algebra and $s^2 W_a(\theta)$ is consistently defined on it. Explicitely, one finds the following equivalent expressions for the square

$$\begin{aligned} s^2(W_a)(\theta) &= T^+(\theta + 2\pi i)_a^m W_n(\theta) sT^+(\theta)_m^n \\ &= W_n(\theta + 2\pi i) sT^+(\theta + 2\pi i)_m^n T^-(\theta + 2\pi i)_a^m \\ &= T^+(\theta + 2\pi i)_a^m sT^-(\theta)_m^n W_n(\theta + 2\pi i) \\ &= sT^-(\theta + 2\pi i)_m^n W_n(\theta + 4\pi i) T^-(\theta + 2\pi i)_a^m . \end{aligned} \quad (2.22)$$

In fact $W_a^-(\theta) := s^2 W_a(\theta - 2\pi i)$ can be viewed as a ‘higher order copy’ of $W_a^+(\theta) := W_a(\theta)$. It again satisfies the relations (TW) and (WW) and

$$C^{mn} W_m^-(\theta) T^-(\theta + i\pi)_n^a = C^{mn} T^+(\theta + i\pi)_n^a W_m^-(\theta + 2\pi i) , \quad (2.23)$$

which differs from (S) by the interchange of $T^+(\theta)$ and $T^-(\theta)$. (So that $W_a(\theta)$ and $s^2 W_a(\theta - 2\pi i)$ can not quite serve to model the generators $W_a^\pm(\theta)$ of $F_\pm(S)$.) In addition one has ‘mixed’ (WW) relations

$$W_a^+(\theta_1) W_b^-(\theta_2) = S_{ab}^{dc}(\theta_{12}) W_c^-(\theta_2) W_d^+(\theta_1) , \quad Re \theta_{12} \neq 0 .$$

Clearly the above process can be iterated. Supplementing the defining relations of $F_*(S)$ by (S^k) , $k \leq n$ the powers $s^k W_a(\theta)$, $k \leq n$ will be well-defined and yield higher order copies of $W_a(\theta)$ and $sW_a(\theta)$. The odd powers commute with the even ones, and the even powers satisfy (TW), (WW) and either (S) or the flipped version (2.23). If no further relations are imposed, this process never leads back to the original generators i.e. $s^{2n}W_a(\theta) \neq W_a(\theta + n\pi i)$, $s^{2n+1}W_a(\theta) \neq sW_a(\theta + n\pi i)$ for all $n > 0$. Truncations can be achieved by imposing extra relations in the $T(S)$ subalgebra. As an example, suppose that the following extra condition is imposed

$$(T3) \quad \begin{aligned} C_{mn}T^+(\theta + i\pi)_a^m T^-(\theta)_b^n &= C_{mn}T^-(\theta + i\pi)_a^m T^+(\theta)_b^n, \\ C^{mn}T^+(\theta + i\pi)_m^a T^-(\theta)_n^b &= C^{mn}T^-(\theta + i\pi)_m^a T^+(\theta)_n^b, \end{aligned}$$

where the first and the second line are related by the application of s . The relation (T3) has a number of implications: First, it implies that the W -generators themselves satisfy the flipped (S) relations (2.23). This can be seen by comparing two different expressions for $s^3W_a(\theta)$. From (2.22) and (T3) one finds

$$sT^+(\theta + 2\pi i)_a^m W_m(\theta + 4\pi i) = s^3W_a(\theta) = W_m(\theta + 2\pi i)sT^-(\theta + 2\pi i)_a^m,$$

which implies (2.23) with $W_a^-(\theta)$ replaced by $W_a(\theta)$. In particular this has the consequence that s has an inverse. Set

$$s^{-1}W_a(\theta) = W_m(\theta - 2\pi i)sT^-(\theta - 2\pi i)_a^m = sT^+(\theta - 2\pi i)_a^m W_m(\theta), \quad (2.24)$$

which is well-defined and can readily be checked to be inverse to s . Comparing with the previous equation then yields $s^3W_a(\theta) = s^{-1}W_a(\theta + 4\pi i)$ i.e.

$$s^4W_a(\theta) = W_a(\theta + 4\pi i), \quad (2.25)$$

which is the truncation announced before.

2.5 Relation of $F_{\pm}(S)$ to real rapidity algebras

The doublet $F_{\pm}(S)$ bears no obvious relation to the Zamolodchikov-Faddeev algebra $Z(S)$, which encodes the algebraic features of a factorized scattering theory. The characteristics of the ZF-algebra are that there are two sets of generators $Z_a(\theta)$, $\overline{Z}^a(\theta)$, both defined for *real* rapidities only. Of course also the additional generators $T^{\pm}(\theta)_a^b$ are lacking. In appendix A we introduce extended ZF-algebras, where the generators $Z_a(\theta)$, $\overline{Z}^a(\theta)$ are

supplemented by generators $T^\pm(\theta)_a^b$ and study the relations (among the T 's and the mixed products $ZT, \bar{Z}T$) that can consistently be imposed in such an extended ZF-algebra. In particular there exists an algebra $TZ(S)$ of that type, which can be regarded as the symmetry algebra of the factorized scattering theory. As a by-product one obtains a proof of the consistency of the algebra $F_*(S)$ and hence of $F_\pm(S)$. Recall from section 2.3 that by means of the relation (S) the numerical residue equations (R \pm) imply operator-valued residue equations (2.10). In order to mimik this effect in a real rapidity algebra, we introduce an algebra $R(S)$, which in a sense interpolates between the quotients of the extended ZF-algebra and the form factor algebra. The generators of $R(S)$ include pairs of $Z_a(\theta)$, $\bar{Z}^a(\theta)$ generators and are defined for $Im\theta \in \pi\mathbb{Z}$ only. In a slight abuse of notation we shall still refer to $R(S)$ as a ‘real rapidity’ algebra. The delta function term in the ZF-algebra becomes operator-valued by means of the replacement

$$4\pi\delta_a^b\delta(\theta_{12})\mathbb{1} \longrightarrow [C^{aa'}L_{a'b}^\pm(\theta_2) - \delta_a^b]2\pi\delta(\theta_{12}) .$$

In appendix A.5 we show that $R(S)$ is a consistently defined associative extension of the ZF-algebra. Moreover, the generators $Z_a(\theta)$, $\bar{Z}^a(\theta)$, $\theta \in i\pi\mathbb{Z} + \mathbb{R}$ can be combined into a single generator $W_a(\theta)$ with complex arguments. The resulting algebra $F_R(S)$ has generators $W_a(\theta)$, $T^\pm(\theta)_a^b$, $\theta \in \mathbb{C}$ and may be viewed as a ‘reduced’ version of a form factor algebra. The crucial simplification is that the pole singularities in (2.15) have been replaced by delta function singularities. Off the singularities $F_R(S)$ is isomorphic to $F_*(S)$. In appendix B we show that $F_R(S)$ can also be considered as arising through a reduction process from an alternative form factor algebra $F(S)$. Symbolically the relations among the various algebras are summarized as follows

$$F(S) \xrightarrow{red.} F_R(S) \simeq R(S) \xrightarrow{red.} TZ(S) \xrightarrow{subalg.} Z(S) .$$

The details of this construction are deferred to the appendix. For the algebraic characterization of form factors described in the next section only the following relation between the generators $Z_a(\theta)$, $\bar{Z}^a(\theta)$, $\theta \in i\pi\mathbb{Z} + \mathbb{R}$ of $R(S)$ and the generators $W_a(\theta)$ of the reduced algebra $F_R(S)$ is needed

$$\begin{aligned} W_a(\theta - i\epsilon) &= \begin{cases} Z_a(\theta) , & \theta \in \mathbb{R} \\ C_{aa'}\bar{Z}^{a'}(\theta - i\pi) , & \theta \in \mathbb{R} + i\pi . \end{cases} \\ W_a(\theta + i\epsilon) &= \begin{cases} C_{aa'}\bar{Z}^{a'}(\theta - i\pi) , & \theta \in \mathbb{R} \\ Z_a(\theta) , & \theta \in \mathbb{R} - i\pi , \end{cases} \end{aligned} \quad (2.26)$$

where the limit $\epsilon \rightarrow 0^+$ is to be taken. Off the singularities this also gives a correspondence between the generators of $F_R(S)$ and $F_*(S)$.

3. Form factors as linear forms over $F_{\pm}(S)$

The Karowski-Weisz-Smirnov axioms for the form factors [1, 2] are conveniently listed as follows

- (1) Relation between In and Out states.
- (2) Exchange relation.
- (3) KMS-property.
- (4) Kinematical residue axiom.
- (5) Bound state residue axiom.
- (6) Inner product.

We shall discuss these axioms consecutively below along with their algebraic implementation. Form factors will be identified as sums of T -invariant linear forms over $F_{\pm}(S)$. Recall from section 2.2 the notion of an analytic linear form over $F_{\pm}(S)$. We call a linear form *T-invariant* if it satisfies in addition

$$f(X T^+(\theta)_a^b) = \delta_a^b f^{\pm}(X) \quad (3.1a)$$

$$f(T^-(\theta)_a^b X) = \omega(a) \delta_a^b f(X) , \quad \theta \in \mathbb{C} , \quad (3.1b)$$

where $X \in F_{\pm}(S)$ has rapidities separated from θ . In the second line $\omega(a)$ is a phase $|\omega(a)| = 1$ that depends only on the functionals f^{\pm} and a but not on X . The extra condition that of Lorentz invariance $f(X K) = 0 = f(K X) = 0$, where K is the generator of Lorentz boosts, will always be understood without further mentioning. For some $X^{\pm} = W_{a_n}^{\pm}(\theta_n) \dots W_{a_1}^{\pm}(\theta_1) \in F_{\pm}(S)$ we shall also write

$$f^{\pm}(X^{\pm}) = f_{a_n \dots a_1}^{\pm}(\theta_n, \dots, \theta_1) , \quad (3.2)$$

which as a function of the rapidity variables may be viewed as the germ of some analytic function. The main result of this section is that, due to the structure of $F_{\pm}(S)$, the (multivalued) analytic functions arising are precisely the form factors:

For every pair of T -invariant linear forms f^{\pm} over $F_{\pm}(S)$, respectively, their sum $f^+ + f^-$ satisfies axioms (2)–(4). The associated sesquilinear form (2.9) contravariant w.r.t. σ satisfies (1).

The dependence on the local operator enters through the specific form of the functionals f^\pm . Under these conditions Smirnov's formula [2, Eqn.(28)] for the inner product (6) just has the status of a definition. Implicit, however, is the statement that the inner product so defined coincides with the physical inner product on the space of scattering states; which is why we included it as part of the axioms. The only axiom not covered is that for the residues of the bound state poles. We expect that also this axiom can be implemented on an algebraic level by supplementing further relations to $F_\pm(S)$. As the complete set of bound state poles is strongly model-dependent, the discussion of axiom (5) is best deferred to case studies.

3.1 Verification of the form factor axioms

Here we discuss the algebraic implementation of axioms (1)–(3) consecutively. It suffices to use T -invariant functionals f over the algebra $F_*(S)$. The inclusion of the kinematical residue axiom is discussed in section 3.3.

(1) Relation between In and Out states: Let $(\theta'_1, \dots, \theta'_m)$, (b_1, \dots, b_m) be the data of some asymptotic ‘out’ state (rapidities and quantum numbers) and similarly $(\theta_n, \dots, \theta_1)$, (a_n, \dots, a_1) that of an ‘in’ state. Let $(F^\mathcal{O})_{a_n \dots a_1}^{b_1 \dots b_m}(\theta'_1, \dots, \theta'_m | \theta_n, \dots, \theta_1)$ denote the matrix element of some local operator $\mathcal{O}(x)$ between these states. The axiom (1) states that this matrix element is related to an $n + m$ particle form factor by means of the relation*

$$\begin{aligned} & (F^\mathcal{O})_{a_n \dots a_1}^{b_1 \dots b_m}(\theta'_1, \dots, \theta'_m | \theta_n, \dots, \theta_1) \\ &= C^{b_1 c_1} \dots C^{b_m c_m} f_{c_1 \dots c_m a_n \dots a_1}^\mathcal{O}(\theta'_1 - i\pi, \dots, \theta'_m - i\pi, \theta_n, \dots, \theta_1), \end{aligned} \quad (3.3)$$

where all rapidities θ_i, θ'_i lie in $\mathbb{R} + i\epsilon$ and are separated. Here a set of rapidities $(\omega_1, \dots, \omega_n)$ is called separated if $|\omega_i - \omega_j| > \delta$, for some $\delta > 0$. Within the algebraic formulation this statement is simply the definition of the sesquilinear form (2.9) contravariant w.r.t. the antilinear anti-involution σ . One has

$$\begin{aligned} & f(W_{c_1}(\theta'_1 - i\pi) \dots W_{c_m}(\theta'_m - i\pi) W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1)) \\ &= f\left(\sigma(W_{c_1})(\theta'_1 + i\pi) \dots \sigma(W_{c_m})(\theta'_m + i\pi) W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1)\right) \\ &= F\left(W_{c_m}(\theta'_m + i\pi) \dots W_{c_1}(\theta'_1 + i\pi), W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1)\right). \end{aligned} \quad (3.4)$$

Since all rapidities are in an $i\epsilon$ -neighbourhood of the real axis and separated one can also to rewrite (3.4) in terms of the generators of the algebra $R(S)$ described in appendix A.

*The axiom is usually formulated for real rapidities only. We consider rapidities in an $i\epsilon$ -neighbourhood of the real axis to emphasize the structure of the underlying involution.

Write

$$F_{a_n \dots a_1}^{b_1 \dots b_m}(\theta_1^*, \dots, \theta_m^* | \theta'_n, \dots, \theta'_1) = F(Y, X) ,$$

where $X = Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1)$, $Y = \overline{Z}^{b_m}(\theta_m^*) \dots \overline{Z}^{b_1}(\theta_1^*)$. Using the correspondence (2.26) for $\theta'_i \in \mathbb{R} + i\epsilon$ equation (3.4) becomes

$$\begin{aligned} F_{a_n \dots a_1}^{b_1 \dots b_m}(\theta_1^*, \dots, \theta_m^* | \theta_n, \dots, \theta_1) &= F(Y, X) = f(\sigma(Y) X) = \\ &= C^{b_1 c_1} \dots C^{b_m c_m} f_{c_1 \dots c_m a_n \dots a_1}(\theta'_1 - i\pi, \dots, \theta'_m - i\pi, \theta_n, \dots, \theta_1) , \end{aligned} \quad (3.5)$$

which is (3.3). The dependence on the local operator $\mathcal{O}(x)$ enters through the specific form of the functional f and will in general be suppressed in the notation. In order to make contact with the usual interpretation of $F(Y, X)$ as a matrix element, let us assume (if only for heuristic reasons) that some analogue of the GNS construction exists for the functionals f . (They are not positive functionals and $F_*(S)$ is not a C^* algebra.) Explicitely, suppose that f can be written as a (bi-) vector functional $f^\mathcal{O}(X) = (|\mathcal{O}\rangle, X|v\rangle)$, where the notation $|\mathcal{O}\rangle := \mathcal{O}(0)|v\rangle$ indicates that the operator $\mathcal{O}(x)$ is supposed to be uniquely specified by its action on the vacuum. The matrix elements and form factors of the local operator $\mathcal{O}(x)$ then are related by

$$F^\mathcal{O}(Y, X) = (Y|\mathcal{O}\rangle, X|v\rangle) = (|\mathcal{O}\rangle, \sigma(Y) X|v\rangle) = f^\mathcal{O}(\sigma(Y) X) . \quad (3.6)$$

In particular one sees that for a nontrivial operator \mathcal{O} one should take $f(\mathbb{1}) = 0$. For the present purposes there is no need to make (3.6) precise. The form factors serve only as a tool to construct the Wightman functions and the usual GNS construction will apply to them. Here equation (3.6) is merely taken to illustrate that, compared to the standard definition, one is dealing with the form factors of the adjoint operator $\mathcal{O}^\dagger(0)$. The phase $\omega(a) = \omega(\mathcal{O}, W_a)$ in (3.1b) represents the relative locality index of $\mathcal{O}(x)$ with the asymptotic fields. The T -invariance of f then just corresponds to the condition that $|v\rangle$ and $|\mathcal{O}\rangle$ are ‘highest weight’ states of the $T(S)$ subalgebra in the sense that

$$\begin{aligned} T^+(\theta)_a^b |v\rangle &= \delta_a^b |v\rangle , \\ \sigma(T^-)_a^b(\theta) |\mathcal{O}\rangle &= \omega(a) \delta_a^b |\mathcal{O}\rangle . \end{aligned} \quad (3.7)$$

We shall refer to such representations as *T-invariant* representations of $F_*(S)$. Note that it is *not* required that $|v\rangle$ and/or $|\mathcal{O}\rangle$ have any vacuum properties w.r.t. the $W_a(\theta)$ generators; generically one will not be dealing with Fock-type representations of $F_*(S)$.

(2) Exchange relation: This is a trivial consequence of the (WW) relations.

$$f_{a_1 \dots a_{i+1}, a_i \dots a_n}(\theta_n \dots, \theta_i, \theta_{i+1}, \dots, \theta_1) = S_{a_{i+1} a_i}^{d c}(\theta_{i+1}, i) f_{a_1 \dots c, d \dots a_n}(\theta_n \dots, \theta_{i+1}, \theta_i, \dots, \theta_1) , \quad (3.8)$$

which is the exchange axiom. Initially this axiom was formulated for real rapidities only. Here it holds for $\text{Re } \theta_{i+1,i} \neq 0$. This extension to generic complex rapidities is non-trivial and is equivalent to a system of linear difference equations (c.f. section 3.2).

(3) KMS property: This axiom prescribes the behaviour of form factors under analytic continuation $\theta \rightarrow \theta + 2\pi i$ of one of the rapidity variables. The condition is

$$f_{a_n \dots a_1}(\theta_n + 2\pi i, \theta_{n-1}, \dots, \theta_1) = \omega(a_n) f_{a_{n-1} \dots a_1 a_n}(\theta_{n-1}, \dots, \theta_1, \theta_n) . \quad (3.9)$$

Initially this axiom again was formulated for real rapidities $\theta_1, \dots, \theta_n$ only. We shall see in section 3.2 that it can be extended to generic complex rapidities. If one considers θ formally (for the time being) as a time variable and assumes $\omega(a_n) = 1$, this is precisely a KMS condition for a thermal (quasi) state of inverse temperature 2π . This analogy can be pushed further[24, 25], but in the present context it just serves to motivate the name ‘KMS property’ for (3.9). By repeated use of the exchange relation (3.8) an equivalent form is

$$\begin{aligned} & f_{a_n \dots a_1}(\theta_n + 2\pi i, \theta_{n-1}, \dots, \theta_1) \\ &= \omega(a_n) S_{a_{n-1} c_{n-1}}^{b_{n-1} b_n}(\theta_{n-1, n}) S_{a_{n-2} c_{n-1}}^{b_{n-2} c_{n-2}}(\theta_{n-2, n}) \dots S_{a_1 a_n}^{b_1 c_1}(\theta_{1, n}) f_{b_n \dots b_1}(\theta_n, \dots, \theta_1) \end{aligned} \quad (3.10)$$

In the algebraic formulation equation (3.10) is a consequence of the relations (S). Using the equivalent form (2.7) one has

$$\begin{aligned} & f(W_{a_n}(\theta_n + 2\pi i) W_{a_{n-1}}(\theta_{n-1}) \dots W_{a_1}(\theta_1)) \\ &= f(T^-(\theta_n + 2\pi i)_{a_n}^n W_m(\theta_n) s(T^+)_n^m(\theta_n) W_{a_{n-1}}(\theta_{n-1}) \dots W_{a_1}(\theta_1)) . \end{aligned} \quad (3.11)$$

In the last line one first applies the vacuum condition (3.1b). Then one pushes $s(T^\pm)_n^m(\theta_n)$ to the right, using (TW) and applies the vacuum condition (3.1a). Comparison yields (3.10). One can also return to the original form (3.9) i.e.

$$f(W_{a_n}(\theta_n + 2\pi i) W_{a_{n-1}}(\theta_{n-1}) \dots W_{a_1}(\theta_1)) = \omega(a_n) f(W_{a_{n-1}}(\theta_{n-1}) \dots W_{a_1}(\theta_1) W_{a_n}(\theta_n)) .$$

That is to say, the T -invariant functionals f , when restricted to strings of $W_a(\theta)$ ’s automatically are KMS functionals. The implementation via (3.11) however is more stringent in that the same T -operators also implement the kinematical residue axiom and satisfy TTS relations. Moreover, an argument analogous to (3.11) leads to a compatible system of linear difference equations (deformed KZE) generalizing (3.10).

3.2 Deformed Knizhnik-Zamolodchikov Equation

Identities similar to (3.10) arise when one evaluates expectation functionals with an insertion of equation (2.7) at the i -th position, using the invariance conditions (3.1a,b). After some rearrangement one finds

$$f_{a_n \dots a_i \dots a_1}(\theta_n, \dots, \theta_i + 2\pi i, \dots, \theta_1) = (A_i)_{a_n \dots a_1}^{b_n \dots b_1}(\theta_n, \dots, \theta_1) f_{b_n \dots b_1}(\theta_n, \dots, \theta_1) , \quad (3.12)$$

valid for generic *complex* rapidities $\theta_n, \dots, \theta_1$. Here

$$\begin{aligned} & (A_i)_{a_n \dots a_1}^{b_n \dots b_1}(\theta_n, \dots, \theta_1) \\ &= \omega(a_i) S_{a_{i+1} a_i}^{b_{i+1} c_{i+1}}(\theta_{i+1, i} - 2\pi i) S_{a_{i+2} c_{i+1}}^{b_{i+1} c_{i+2}}(\theta_{i+2, i} - 2\pi i) \dots S_{a_n c_{n-1}}^{b_n c_i}(\theta_{n, i} - 2\pi i) \times \\ & \times S_{a_1 c_i}^{b_1 c_2}(\theta_{1, i}) S_{a_2 c_2}^{b_2 c_3}(\theta_{2, i}) \dots S_{a_{i-1} c_{i-1}}^{b_{i-1} b_i}(\theta_{i-1, i}) \\ &= \left[\omega(a_i) S_{i, i+1}(\theta_{i, i+1} + 2\pi i)^{-1} S_{i, i+2}(\theta_{i, i+2} + 2\pi i)^{-1} \dots S_{i, n}(\theta_{i, n} + 2\pi i)^{-1} \times \right. \\ & \left. \times S_{1, i}(\theta_{1, i}) S_{2, i}(\theta_{2, i}) \dots S_{i-1, i}(\theta_{i-1, i}) \right]_{a_n \dots a_1}^{b_n \dots b_1} , \end{aligned} \quad (3.13)$$

using the standard matrix notation in the last line. Introducing the shift operators $(T_i f)(\theta_n, \dots, \theta_1) = f(\theta_n, \dots, \theta_i + 2\pi i, \dots, \theta_1)$ equation (3.12) becomes a system of linear difference equations

$$T_i f = A_i f , \quad 1 \leq i \leq n , \quad (3.14)$$

of the form studied in [19] ('deformed Knizhnik-Zamolodchikov Equations' (KZE)). As such, the operators A_i must satisfy the compatibility conditions

$$(T_i A_j) A_i = (T_j A_i) A_j . \quad (3.15)$$

This is indeed the case, the computation of [19, Theorem 5.2] carries over. Observe also that the KZE imply

$$f_{a_n \dots a_1}(\theta_n + 2\pi i, \dots, \theta_1 + 2\pi i) = f_{a_n \dots a_1}(\theta_n, \dots, \theta_1) . \quad (3.16)$$

The relevance of the deformed KZE to form factors was first pointed out by Smirnov [9], although no derivation was given. The system of compatible equations (3.14) is much stronger than the KMS condition (3.9), which corresponds to the case $i = n$ and *real* rapidities $\theta_n, \dots, \theta_1$. Initially both, the exchange relations (3.8) and the KMS condition (3.9), were introduced for real rapidities only. If one formally applies (3.8) also for generic

complex rapidities (and assumes that $\omega(a_i)$ is independent of i), the deformed KZE can be seen to be equivalent to (3.10), now valid for generic complex rapidities. Thus, in upshot, what the deformed KZE tell is that on the level of the form factors one can extend both, the exchange relations and the KMS property, to generic complex rapidities. Implicit in this extension there are strong consistency conditions, which are made manifest in (3.15). In an algebraic formulation the main problem is to reconcile (WW) exchange relations for complex rapidities $W_a(\theta_1)W_b(\theta_2) = S_{ab}^{dc}(\theta_{12})W_c(\theta_2)W_d(\theta_1)$ with the implementation of the kinematical residue axiom (Eqn. (3.17) below). Formally one can achieve this by postulating Smirnov's operator-valued residue equation (2.15). If one then extends the quantum double by antiunitary operators U_π, U_π^\dagger that act on the local operator \mathcal{O} and considers the In-Out axiom (1) as being given, one can also arrive at the KZE[6]. In the present formulation the only extra ingredient is the relation (S), which implies both, the KZE and kinematical residue equations via (R \pm) and (2.10). The latter is detailed in the next section.

3.3 Kinematical residue axiom

This axiom states that form factors $f_{a_n \dots a_1}(\theta_n, \dots, \theta_1)$ have simple poles at relative rapidities $i\pi$ with prescribed residues. Explicitly

$$\begin{aligned}
& 2\pi i \operatorname{res} f_{a_n \dots a_{i+1} a_i \dots a_1}(\theta_n, \dots, \theta_i + i\pi, \theta_i, \dots, \theta_1) \\
&= f_{b_n \dots b_{i+2} b_{i-1} \dots b_1}(\theta_n, \dots, \theta_{i+2}, \theta_{i-1} \dots, \theta_1) C_{a_{i+1}c} \times \\
&\times \left\{ \omega(a_{i+1}) S_{c_{i+1}a_n}^{c_{n-1}b_n}(\theta_{in} + 2\pi i) S_{c_{n-1}a_{n-1}}^{c_{n-2}b_{n-1}}(\theta_{in-1} + 2\pi i) \dots S_{c_{i+2}a_{i+2}}^{c_{i+1}b_{i+2}}(\theta_{ii+2} + 2\pi i) \times \right. \\
&\times \left. S_{a_i a_{i-1}}^{c_{i-1}b_{i-1}}(\theta_{ii-1}) S_{c_{i-1}a_{i-1}}^{c_{i-2}b_{i-2}}(\theta_{ii-2}) \dots S_{c_2 a_1}^{c_{i+1}b_1}(\theta_{i1}) - \delta_{a_i}^c \delta_{a_n}^{b_n} \dots \delta_{a_{i+2}}^{b_{i+2}} \delta_{a_{i-1}}^{b_{i-1}} \dots \delta_{a_1}^{b_1} \right\}. \quad (3.17)
\end{aligned}$$

We shall make use of the following reformulation of the kinematical residue axiom. Let $f_{a_n \dots a_1}^\pm(\theta_n, \dots, \theta_1)$ be solutions of the form factors axioms (1) – (3), which instead of (3.17) satisfy the simpler residue conditions

$$\begin{aligned}
& 2\pi i \operatorname{res} f_{a_n \dots a_{i+1} a_i \dots a_1}^\pm(\theta_n, \dots, \theta_i \mp i\pi, \theta_i, \dots, \theta_1) \\
&= -C_{a_{i+1}a_i} f_{b_n \dots b_{i+1} b_{i-2} \dots b_1}^\pm(\theta_n, \dots, \theta_{i+2}, \theta_{i-1} \dots, \theta_1). \quad (3.18)
\end{aligned}$$

Then

$$f_{a_n \dots a_1}(\theta_n, \dots, \theta_1) := f_{a_n \dots a_1}^+(\theta_n, \dots, \theta_1) + f_{a_n \dots a_1}^-(\theta_n, \dots, \theta_1) \quad (3.19)$$

satisfies (3.17) and hence all of the axioms (1) – (4). This is a consequence of the KZE. Rewrite the f^- term as

$$f_{a_n \dots a_1}^-(\theta_n, \dots, \theta_{i+1}, \theta_i, \dots, \theta_1) \\ = \omega(a_{i+1})(A_{i+1})_{a_n \dots a_1}^{b_n \dots b_1}(\theta_n, \dots, \theta_{i+1} - 2\pi i, \theta_i, \dots, \theta_1) f_{a_n \dots a_1}^-(\theta_n, \dots, \theta_{i+1} - 2\pi i, \theta_i, \dots, \theta_1).$$

Since[†]

$$(A_{i+1})_{a_n \dots a_1}^{b_n \dots b_1}(\theta_n, \dots, \theta_i - \pi i, \theta_i, \dots, \theta_1) C_{b_{i+1} b_i} \\ = -S_{c_{i+1} a_n}^{c_{n-1} b_n}(\theta_{in} + 2\pi i) S_{c_{n-1} a_{n-1}}^{c_{n-2} b_{n-1}}(\theta_{in-1} + 2\pi i) \dots S_{c_{i+2} a_{i+2}}^{c_{i+1} b_{i+2}}(\theta_{ii+2} + 2\pi i) \times \\ \times S_{a_i a_{i-1}}^{c_{i-1} b_{i-1}}(\theta_{ii-1}) S_{c_{i-1} a_{i-1}}^{c_{i-2} b_{i-2}}(\theta_{ii-2}) \dots S_{c_2 a_1}^{c_{i+1} b_1}(\theta_{i1}), \quad (3.20)$$

the claim follows. Within the algebraic framework we have seen that any T -invariant functional f^+ or f^- over $F_+(S)$ or $F_-(S)$, respectively, automatically satisfies the form factor axioms (1) – (3). Thus, given the residue condition (R \pm) for the $W_a^\pm(\theta)$ generators, the functions (3.2) will satisfy (3.18), so that their sum satisfies the kinematical residue equation. As an aside we remark that the latter also follows directly from Smirnov's residue prescription (2.15)

$$2\pi i \operatorname{res} f_{a_n \dots a_{i+1} a_i \dots a_1}(\theta_n, \dots, \theta_i + i\pi, \theta_i, \dots, \theta_1) \\ = f(W_{a_n}(\theta_n) \dots W_{a_{i+2}}(\theta_{i+2}) [L_{a_i a_{i+1}}^+(\theta_i) - C_{a_i a_{i+1}}] W_{a_{i-1}}(\theta_{i-1}) \dots W_{a_1}(\theta_1)) \quad (3.21)$$

Inserting $L_{ab}^+(\theta) = C_{mn} T^-(\theta + i\pi)_a^m T^+(\theta)_b^n$ and using the T -invariance conditions (3.1), equations (3.17) is easily reproduced by means of the (TW) exchange relations. For the reasons explained in section 2.4 we prefer the first derivation.

In summary, we have shown that the algebraic structure of the form factor algebra $F(S)$ encodes all the form factor axioms (1)–(4) in the sense stated after equation (3.1). Axiom (6) then has the status of a definition and the inclusion of bound state poles is a separate issue.

4. Diagonalization of the conserved charges

The form factor algebra contains a subalgebra $T(S)$ of ‘ TTR -type’. By an algebra of TTR -type we mean any associative algebra with one or more sets of generators $T(\theta)_a^b$ subject to the relations

$$R_{ab}^{mn}(\theta_{12}) T(\theta_2)_n^d T(\theta_1)_m^c = T(\theta_1)_a^m T(\theta_2)_b^n R_{mn}^{cd}(\theta_{12}),$$

[†]Note that the contraction $S_{ab}^{dc}(-i\pi)C_{dc} = C_{aa'}S_{bc}^{ca'}(2\pi i) = -C_{ab}$ is regular, even if $S_{ab}^{dc}(-i\pi)$ is not.

where the numerical R -matrix $R_{ab}^{dc}(\theta_{12})$ characterizes the algebra (similar to the structure constants of a Lie algebra). Such algebras contain an abelian subalgebra $[t(\theta), t(\theta')] = 0$, $\theta \neq \theta'$ generated by the traces $t(\theta) := T(\theta)_a^a$. One is interested in representations where this abelian subalgebra is diagonalizable. In the context of integrable relativistic QFTs the basic proposal is that for the ‘correct’ R -matrix and the ‘correct’ representation the eigenstates of $t(\theta)$ can be interpreted as the asymptotic multiparticle states and the eigenvalues are generating functions for the spectrum of the local conserved charges on these states. The problem to be solved is:

What R -matrix and what representations are relevant for the description of a given integrable QFT? ()*

The remarkable achievement of the QISM is that in many cases it provides a technique to construct both, the R matrix and the dynamically correct representation, starting from the classical theory. As indicated in the introduction there are however classes of integrable QFTs where this technique does not apply. Since the form factor algebra contains a subalgebra of $T\bar{T}R$ -type it seems natural to use the relation to form factors to propose an alternative solution to the problem (*). In this section we shall formulate such a criterion and use it to derive a formula for the eigenvalues of the local conserved charges valid for any QFT with a mass gap and diagonal factorized scattering theory.

4.1 How to select the dynamically correct representation of the $T\bar{T}R$ algebra

“The determination of the representation class of the canonical commutation relations is a dynamical problem” [31, p.57] At least in an abstract sense this is the source of much of the nontrivial features of QFT. Models of the Wightman axioms, it seems, cannot be defined directly, but only through a limit of ‘regularized’ auxiliary systems, each of which violates one or more of the axioms. One is forced to adopt such painfully indirect procedures because the representations of the canonical commutation relations that are compatible with some non-trivial interaction are not of Fock-type (‘Haags theorem’), but are inaccessible otherwise.

In integrable QFTs specifically, this generic problem has a nonlinear counterpart: The determination of the representation class of the $T\bar{T}R$ algebra is a dynamical problem. The analogy between both problems has first been pointed out by Sklyanin [14] in the course of generalizing the algebraic Bethe Ansatz. The motivation to search for such a generalization stems from the following consideration. The QISM answers the questions (*) in the following way: The ‘correct’ R -matrix is the one obtained by q -deforming the classical r -matrix which appears in the Poisson brackets of the spatial component of the

linear system. Given this R -matrix one can construct an integrable lattice model in finite volume in which the traces $t_{a,L}(\theta)$ (a and L indicating the UV and volume cutoff, respectively) can be diagonalized by means of the algebraic Bethe Ansatz. In general however the cutoffs cannot be removed without running into singularities, which usually reflects the non-ferromagnetic nature of the physical vacuum. In the representation theoretical language used before, this means that the infinite dimensional representation of the $T\bar{T}R$ algebra, which one tries to construct as a limit of the finite dimensional representations in the cutoff theory, is not (the) one which is compatible with the dynamics. Roughly speaking, the idea of the functional Bethe Ansatz [14, 13] is to work directly with the cutoff-free continuum theory and to find the dynamically correct infinite dimensional representation by ‘educated guess’. The amount of guesswork required depends on the type of model considered. For models with a ferromagnetic vacuum it is fairly small and the functional Bethe Ansatz is a genuine alternative to the QISM in the usual sense. For models where the physical vacuum is not ferromagnetic, the amount of guesswork is considerable[13].

Here we propose a criterion how to select the dynamically correct infinite dimensional representation of the $T\bar{T}R$ -algebra. All the guesswork is concentrated in finding the excitation spectrum of the model and the associated two-particle bootstrap S -matrix. As a matter of fact this is usually much simpler than to set up the machinery of (some version of) the QISM. Moreover, compared to the situation in the late 70s, when the QISM was developed[3], there are meanwhile several non-perturbative techniques to test proposed bootstrap S -matrices. In particular, the thermodynamic Bethe Ansatz [28] and Monte Carlo simulations [27] may count as such. Given a reliable candidate for a bootstrap S -matrix one can build the algebra $F_{\pm}(S)$ with subalgebra $T(S)$ of $T\bar{T}R$ -type, where R is the physical S -matrix. The proposed criterion how to select the dynamically correct infinite dimensional representation of the $T\bar{T}R$ (here: $T\bar{T}S$)-algebra then simply is

Criterion: Select those representations of $T(S)$ that can be extended to a T -invariant representation of $F_{\pm}(S)$.

The motivation is as follows. By construction, any bilinear form defined on a pair of such representations will automatically solve all the form factor axioms. The choice of a specific representation satisfying the criterion reflects the choice of a specific local operator. By the design of the form factor axioms, the resulting Wightman functions have all the desired properties; in particular they satisfy locality[2]. In other words, the ‘dynamical correctness’ is expected to be built in. On the other hand the criterion is also strong enough to determine the spectrum of the conserved charges at least when the scattering theory is diagonal.

4.2 Diagonal FST: Spectrum of maximal abelian subalgebra

Let a factorized scattering theory be given with a diagonal S -matrix $S_{ab}^{dc}(\theta) = S_{ab}(\theta)\delta_a^d\delta_b^c$ satisfying the bootstrap equations (displayed at the end of section A.1). Consider the form factor algebra specialized to this case. (T1) becomes

$$(T1) \quad t_a^\pm(\theta_1) t_b^\pm(\theta_2) = t_b^\pm(\theta_2) t_a^\pm(\theta_1) , \quad t_a^\pm(\theta_1) t_b^\mp(\theta_2) = t_b^\mp(\theta_2) t_a^\pm(\theta_1) ,$$

where

$$T^\pm(\theta)_a^b = t_a^\pm(\theta) \delta_a^b , \quad (\text{no sum}) . \quad (4.1)$$

That is to say, $T(S)$ degenerates into a direct product of abelian algebra generated by the diagonal elements of $T^\pm(\theta)_a^b$. Moreover $T(S)$ essentially becomes independent of S . Accordingly the relations (T2) simplify.

$$(T2) \quad t_a^\pm(\theta + i\pi) t_a^\pm(\theta) = \mathbb{1} ,$$

using $C_{ab} = \delta_{a\bar{b}}$. The relations (TW) and (S) become

$$(TW) \quad t_a^\pm(\theta_0) W_b(\theta_1) = S_{ab}(\theta_{01}) W_b(\theta_1) t_a^\pm(\theta_0) ,$$

$$(S) \quad t_a^\pm(\theta) W_{\bar{a}}(\theta - i\pi) = W_{\bar{a}}(\theta + i\pi) t_a^\mp(\theta) .$$

The (WW) relations read

$$(WW) \quad W_a(\theta_1) W_b(\theta_2) = S_{ab}(\theta_{12}) W_b(\theta_2) W_a(\theta_1) .$$

The residue conditions (R \pm) are unchanged; the reversed residue equations (2.10) become

$$\begin{aligned} 2\pi i \operatorname{res}[W_a^+(\theta + i\pi) W_b^+(\theta)] &= \delta_{\bar{a}b} e_b(\theta) , \\ 2\pi i \operatorname{res}[W_a^-(\theta - i\pi) W_b^-(\theta)] &= \delta_{\bar{a}b} e_b(\theta - i\pi) , \end{aligned} \quad (4.2)$$

where $e_a(\theta) := t_{\bar{a}}^-(\theta + i\pi) t_a^+(\theta)$ can be checked to be a Casimir operator of $F_*(S)$. Nevertheless the representations of interest are *not* the ones on which $e_a(\theta)$ assumes a constant value. The T -invariant functionals satisfy

$$f(X t_a^+(\theta)) = f(X) , \quad f(t_a^-(\theta) X) = \omega(a) f(X) , \quad \theta \in \mathbb{C} , \quad (4.3)$$

for $X \in F(S)$ with rapidities separated from θ . Setting $e_a(\theta)$ equal to ± 1 amounts to the reduction $t_a^+(\theta) = \pm t_a^-(\theta)$, which is not the case of interest in the context of form factors. The fact that $e_a(\theta)$ commutes with the W generators is visible on the level of

form factors, although in a more indirect way. The general result described in section three that $F_{\pm}(S)$ implements the form factor axioms of course carries over to the diagonal case. In particular, for the kinematical residue axiom one finds

$$\begin{aligned} & 2\pi i \operatorname{res} f_{a_n \dots a_{k+1} a_k \dots a_1}(\theta_n \dots \theta_k + i\pi, \theta_k, \dots \theta_1) \\ &= \delta_{\bar{a}_{k+1}, a_k} \left[\prod_{j \neq k, k+1} S_{a_k a_j}(\theta_{kj}) - 1 \right] f_{a_n \dots a_{k+2} a_{k-1} \dots a_1}(\theta_n \dots \theta_{k+2}, \theta_{k-1}, \dots \theta_1) . \end{aligned} \quad (4.4)$$

The fact that all $j \neq k, k+1$ enter on an equal footing is easily seen to be related to $e_a(\theta)$ being central.

Consider now the eigenstates of the abelian algebra $T(S)$ on a T -invariant representation of $F_*(S)$. From the (TW) relations it follows that these are simply strings of W -generators and the spectrum is multiplicative

$$t_a^+(\theta) W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1) |v\rangle = \left(\prod_{i=1}^n S_{aa_i}(\theta - \theta_i) \right) W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1) |v\rangle , \quad (4.5)$$

where all rapidities are supposed to be separated. In accordance with the general principles outlined in the previous section one may thus identify the states in (4.5) with the asymptotic multiparticle states of the theory. On the other hand let $I^{(n)}$, $n \in E$ be the local conserved charges in involution. The integers $n \in E \subset \mathbb{N}$ can be assumed to coincide with the Lorentz spin of the charges they are labeling. By definition these charges have an additive spectrum and one may define the eigenvalues on a n -particle state by

$$I^{(n)} W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1) |v\rangle = \left(\sum_i I^{(n)}(a_i) e^{\pm \theta_i n} \right) W_{a_n}(\theta_n) \dots W_{a_1}(\theta_1) |v\rangle , \quad (4.6)$$

where the sign option is specified below. Comparing eqn.s (4.5) and (4.6) one expects the logarithm of $t_a^+(\theta)$ to be a generating functional for the conserved charges. Of course the meaning of the logarithm first has to be specified. Due to the natural grading by the particle number this is unproblematic. On the n -particle sector $t_a(\theta)$ acts as a finite dimensional (diagonal) matrix and one can define the logarithm of $t_a(\theta)$ simply through the logarithm of this matrix representation. The relation between $\ln t_a(\theta)$ and the conserved charges then is

$$\ln t_a^+(\theta) = \pm i \sum_{n \in E} \left(\frac{e^{\pm \theta}}{c} \right)^n I^{(n)}(a) I^{(n)} , \quad (4.7)$$

where $\pm \operatorname{Re}(\theta - \theta_i) > 0$, acting on a multiparticle state with rapidities $\theta_1, \dots, \theta_n$ and c is a normalization constant. In order to check that both sides of (4.7) have the same action on

multiparticle states, a series expansion of $\ln S_{ab}(\theta)$ is required. It is convenient to split off the sign factor in (2.5) and write $S_{ab}(\theta) = \epsilon_{ab} e^{i\delta_{ab}(\theta)}$. On general grounds the (redefined) scattering phase $\delta_{ab}(\theta)$ admits an expansion

$$\delta_{ab}(\theta) = \pm \sum_{n>0} \frac{d_{ab}(n)}{n} e^{\mp n\theta} , \quad \pm \operatorname{Re} \theta > 0, \quad 0 \leq \operatorname{Im} \theta < s_0 , \quad (4.8)$$

where s_0 is the position of the first bound state pole (i.e. $S_{ab}(is)$ is analytic for $0 < s < s_0$). The S -matrix bootstrap equations, equation (A.7) in particular, put constraints on the expansion coefficients $d_{ab}(n)$. Besides symmetry $d_{ab}(n) = d_{ba}(n)$ one finds

$$\begin{aligned} d_{ab}(n) &= (-)^{n+1} d_{a\bar{b}}(n) , \\ d_{ea}(n) e^{i\eta(a)} + d_{eb}(n) e^{i\eta(b)} + d_{ec}(n) e^{i\eta(c)} &= 0 . \end{aligned} \quad (4.9)$$

The most prominent (possibly all) solutions are those descending from Lie algebraic data. In that case the particles $a = 1, \dots, r$ are associated with the Dynkin diagram of a simple Lie algebra g and the possible fusing angles are selected by the condition

$$\sum_{l=a,b,c} e^{\pm i s \eta(l)} q_l^{(s)} = 0 , \quad (4.10)$$

where $(q_1^{(s)}, \dots, q_r^{(s)})^T$ is the (normalized) eigenvector of the Cartan matrix with eigenvalue $2(1 - \cos \frac{s\pi}{h})$ (h : Coxeter number, s : exponent). The coefficients $d_{ab}(n)$ then take the form

$$d_{ab}(n) = d_n q_a^{(n)} q_b^{(n)} , \quad (4.11)$$

for real constants d_n . The equations (4.9) are satisfied by means of $q_a^{(n)} = (-)^{n+1} q_a^{(n)}$ and (4.10), respectively. Notice that the coefficients (4.11) vanish unless n is an exponent of g modulo h , so that the summations over $n \in \mathbb{N}$ can be replaced by summations over $n \in E$, where E is the set of affine exponents. Both sides of (4.7) then indeed have the same action on multiparticle states, provided one identifies

$$I^{(n)}(a) = c^{n/2} \sqrt{\frac{d_n}{n}} q_a^{(n)} , \quad n \in E , \quad (4.12)$$

using the symmetry $S_{ab}(\theta) = S_{ba}(\theta)$. Equation (4.12) provides a universal formula for the eigenvalues of the conserved charges in any QFT with a mass gap and diagonal factorised scattering theory. As mentioned in the introduction, in QFTs of that type the diagonalization techniques based on the algebraic Bethe Ansatz do not apply. Here the

result is based entirely on the properties of the form factor algebra. Compared to the functional Bethe Ansatz no guesswork is required beyond what is needed to find a reliable candidate for the bootstrap S -matrix. In the case of real coupling affine Toda theories the formula (4.12) reads explicitly [15]

$$I^{(n)}(a) = \left(\frac{m^2 e^T}{2\beta^2} \right)^{n/2} \beta \left[\frac{h \sin \frac{\pi n}{2h} B \sin \frac{\pi n}{2h} (2 - B)}{4\pi n B \sin \frac{\pi n}{h}} \right]^{1/2} q_a^{(n)}, \quad n \in E. \quad (4.13)$$

Here h is the Coxeter number, β is the (bare) coupling constant and $B = \frac{\beta^2/2\pi}{1+\beta^2/4\pi}$ is a nonperturbative effective coupling. The tadpole function T is the sum of all connected vacuum diagrams. It depends on the choice of the renormalization scheme, but the combination $m^2 e^T$ appearing in (4.13) can be seen to be invariant under the normal ordering renormalization group. In particular the parameter c in (4.12) is identified as

$$c = \frac{m^2 e^T}{2\beta^2}. \quad (4.14)$$

A realization of the form factor algebra (for diagonal factorized scattering theories) in terms of the conserved charges and its relation to trace functionals is discussed in [25].

5. Conclusions

The motivation for introducing an algebraic approach to form factors has already been outlined in the introduction. In particular it yields a novel diagonalization technique for the conserved charges, independent of, and alternative to, the QISM. This has been elaborated for the case of QFTs with diagonal factorized scattering theory and we intend to treat the non-diagonal case elsewhere. From the viewpoint of form factors it remains to be seen to what extent the algebra $F_{\pm}(S)$ facilitates the explicit construction of form factors. Since a considerable body of knowledge has been accumulated for the deformed KZE, their algebraic implementation within $F_{\pm}(S)$ should allow one to investigate the subclass of solutions corresponding to form factors. From a conceptual viewpoint, finally, it would be interesting to see whether the analogy between the selection of the dynamically correct representations of the $T\bar{T}R$ -, and that of the canonical commutation relations can be turned into a correspondence. If so, the unfavourable conclusion usually drawn from Haag's theorem could be circumvented in the case of integrable QFTs.

Appendix

A. The extended ZF-algebra and its ideals

In this appendix we study the relation of the form factor algebra $F_{\pm}(S)$ to the Zamolodchikov-Faddeev (ZF)-algebra. The ZF-algebra has two sets of generators $Z_a(\theta)$ and $\overline{Z}^a(\theta)$, both defined for real rapidities only. We shall supplement these generators by operators $T^{\pm}(\theta)_a^b$ having linear exchange relations with the ZF-operators. The resulting ‘extended ZF-algebra’ contains non-trivial ideals. The structure of these ideals determines the relations (among the T ’s and the mixed products $ZT, \overline{Z}T$) that can consistently be imposed on the enlarged set of generators, by switching to the appropriate quotient algebra. In section A.4 examples of such consistent quotient algebras are discussed. As a by-product one obtains a consistency proof for the algebra $F_*(S)$, and hence of $F_{\pm}(S)$. In section A.5 we study an algebra $R(S)$ that can be viewed as a simplified version of a form factor algebra, in which the pole singularities in (2.15) are replaced by delta function singularities. Off the singularities the generators of $R(S)$ can be set into correspondence to that of $F_*(S)$. This correspondence (2.26) is used in the discussion of the form factor axiom (1) in section 3.

A.1 The Zamolodchikov-Faddeev algebra

The Zamolodchikov-Faddeev algebra [30, 3] is an associative algebra with generators $Z_a(\theta)$, $\overline{Z}^a(\theta)$, $\theta \in \mathbb{R}$ a unit $\mathbb{1}$ and the generators P_{μ} , $\epsilon_{\mu\nu}K$ of the 1+1 dimensional Poincaré algebra. The operators $Z_a(\theta)$, $\overline{Z}^a(\theta)$ transform as scalars under the action of the Poincaré group. The defining relations are

$$Z_a(\theta_1) Z_b(\theta_2) = S_{ab}^{dc}(\theta_{12}) Z_c(\theta_2) Z_d(\theta_1) \quad (\text{A.1a})$$

$$\overline{Z}^a(\theta_1) \overline{Z}^b(\theta_2) = S_{dc}^{ab}(\theta_{12}) \overline{Z}^c(\theta_2) \overline{Z}^d(\theta_1) \quad (\text{A.1b})$$

$$\overline{Z}^a(\theta_1) Z_b(\theta_2) = S_{db}^{ac}(\theta_{21}) Z_c(\theta_2) \overline{Z}^d(\theta_1) + 4\pi\delta(\theta_{12})\delta_b^a \mathbb{1} \quad (\text{A.1c})$$

$$Z_a(\theta_1) \overline{Z}^b(\theta_2) = S_{ac}^{db}(\theta_{21} + 2\pi i) \overline{Z}^c(\theta_2) Z_d(\theta_1) + 4\pi\delta(\theta_{12})\delta_a^b \mathbb{1} , \quad (\text{A.1d})$$

where $\theta_{12} = \theta_1 - \theta_2$ etc. and $\delta(\theta)$ is the real delta distribution. We again use Penrose’s abstract index notation [16]. The tensor $S_{ab}^{dc}(\theta)$ is subject to a number of consistency relations. Associativity enforces the Yang Baxter equation. Consistency upon iteration impose the unitarity conditions (2.2) and $S_{an}^{nb}(2\pi i) = -\delta_a^b$, which follows from crossing

invariance and (2.5). In addition we require real analyticity (2.3), while the condition (2.4) is not needed momentarily. For any solution S of the Yang Baxter equation subject to the relations (2.2), (2.3), (2.5) we call the associative algebra (A.1) the Zamolodchikov-Faddeev algebra $Z(S)$ associated with S .

One can formally* also consider the algebra (A.1) with complex rapidities. Let $Z(S)_C$ denote the ‘complexified’ algebra. The $Z(S)_C$ algebra admits a one parameter family of antilinear anti-involutions

$$\begin{aligned}\sigma : Z(S)_C &\longrightarrow Z(S)_C , \\ \sigma^2 &= id , \quad \sigma(z) = z^* , \quad z \in \mathbb{C} , \\ \sigma(XY) &= \sigma(Y)\sigma(X) ,\end{aligned}$$

defined by

$$\begin{aligned}\sigma_\beta(Z_a)(\theta) &= C_{aa'} \overline{Z}^{a'}(\theta^* - i\beta) , \\ \sigma_\beta(\overline{Z}^a)(\theta) &= C^{aa'} Z_{a'}(\theta^* - i\beta) , \quad \beta \in \mathbb{R}\end{aligned}\tag{A.2}$$

This follows from (2.3) and the convention $\sigma[S_{ab}^{dc}(\theta)] = [S_{ba}^{cd}(\theta)]^*$ which adheres to the abstract index notation. In addition there is a linear involution ω given by

$$\begin{aligned}\omega(Z_a)(\theta) &= C_{aa'} \overline{Z}^{a'}(\theta - i\pi) , \\ \omega(\overline{Z}^a)(\theta) &= C^{aa'} Z_{a'}(\theta + i\pi) ,\end{aligned}$$

which is compatible with σ_β in the sense that $(\sigma_\beta \omega)(Z(S)) = (\omega \sigma_\beta)(Z(S))$.

Consider now a $Z(S)_C$ -module Σ which is highest weight in the following sense. There exists a vector $|v_\beta\rangle \in \Sigma$ s.t.

$$\sigma_\beta(Z_a)(\theta)|v_\beta\rangle = 0 \iff \overline{Z}^a(\theta - i\beta)|v_\beta\rangle = 0 , \quad \theta \in \mathbb{R} .$$

Given the antilinear anti-involution σ_β on $Z(S)$ one can define a sesquilinear form $(\ , \)_\beta : \Sigma \times \Sigma \rightarrow \mathbb{C}$ contravariant w.r.t. it, i.e.

$$(Y|v\rangle , X|v\rangle)_\beta = (|v\rangle , \sigma_\beta(Y)X|v\rangle)_\beta .\tag{A.3}$$

The evaluation is done by means of the exchange relations

$$\sigma_\beta(Z_a)(\theta_1^*) Z_b(\theta_2) = S_{ba}^{cd}(\theta_{12} + i\pi - i\beta) Z_c(\theta_2) \sigma_\beta(Z_d)(\theta_1^*) + 4\pi C_{ab} \delta(\theta_{12} - i\beta) .$$

* $Z(S)_C$ has no significance in the context of form factors, for example because of the delta function-, rather than pole-singularities.

Note however that this sesquilinear form is defined uniquely (up to a factor) only for real rapidities. For real rapidities one finds in particular

$$(Z_{b_m}(\omega_m) \dots Z_{b_1}(\omega_1)|v\rangle, Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1)|v\rangle)_\beta = \delta_{nm} \prod_{i=1}^n C_{a_i b_i} 4\pi \delta(\theta_i - \omega_i - i\beta),$$

for $\omega_m > \dots > \omega_1, \quad \theta_n > \dots > \theta_1$.

(A.4)

The matrix elements for other relative orderings of rapidities can be found from (A.1a). Clearly the same matrix elements can be described in terms of the dual highest weight module $\bar{\Sigma}$ based on a state $|\bar{v}\rangle = |\bar{v}_\beta\rangle$ satisfying

$$Z_a(\theta)|\bar{v}_\beta\rangle = 0 \iff \sigma_\beta(\bar{Z}^a)(\theta - i\beta)|\bar{v}_\beta\rangle = 0, \quad \theta \in \mathbb{R}.$$

The evaluation is then done by means of

$$Z_a(\theta_1) \sigma_\beta(Z_b)(\theta_1^*) = S_{ab}^{dc}(\theta_{12} - i\pi + i\beta) \sigma_\beta(Z_c)(\theta_2^*) Z_d(\theta_1) + 4\pi C_{ab} \delta(\theta_{12} + i\beta).$$

In fact, provided S satisfies the ‘Bose symmetry’ (2.4), one can consistently identify

$$(Y|v\rangle, X|v\rangle)_\beta = (\sigma(Y^*)|\bar{v}\rangle, \sigma(X^*)|\bar{v}\rangle)_{2\pi-\beta}, \quad (A.5)$$

where on the r.h.s. $\sigma = \sigma_{2\pi-\beta}$ and X^* equals X except that the rapidities are replaced with their complex conjugates.

Let Σ be the state space of 1+1 dim. relativistic QFT and let Ω^\pm be the bijections onto the asymptotic in/out spaces (Møller operators)

$$\begin{aligned} \Omega^+ : \Sigma &\longrightarrow \Sigma_{in}, \\ \Omega^- : \Sigma &\longrightarrow \Sigma_{out}. \end{aligned}$$

In general $\Sigma_{in/out}$ are proper subspaces of Σ (because of bound states) and assuming weak asymptotic completeness, both are isomorphic and are related by the scattering operator

$$\mathcal{S} = \Omega^+(\Omega^-)^{-1} : \Sigma_{in} \longrightarrow \Sigma_{out}.$$

The spaces $\Sigma_{in/out}$ are Fock spaces graded by the number operator $\Sigma_{in/out} = \bigoplus_{n \geq 0} \Sigma_{in/out}^{(n)}$; where $\Sigma_{in/out}^{(n)}$ are the n -particle subspaces. In the absence of particle production the S -operator preserves the grading $\mathcal{S} : \Sigma_{in}^{(n)} \rightarrow \Sigma_{out}^{(n)}$. In an integrable QFT this n -particle scattering operator factorizes into a product of two particle ones. In terms of the two particle S -matrix one can define the Zamolodchikov-Faddeev algebra $Z(S)$ and describe the entire scattering theory in terms of this algebra. (S, Σ) is called a *factorised scattering theory* if

- $\Sigma_{in/out}$ are highest weight modules of $Z(S)$ with parameter $\beta = 0$. Explicitly, there exists a vector $|v\rangle = |v_0\rangle \in \Sigma^{(0)} = \Sigma_{in/out}^{(0)}$ s.t. $\bar{Z}^a(\theta)|v\rangle = 0$, $\theta \in \mathbb{R}$, where $\mathbb{1}$, $Z_a(\theta)$, $\bar{Z}^a(\theta)$, $\theta \in \mathbb{R}$ are the generators of the $Z(S)$ algebra. The inner product is given by

$$\begin{aligned} & \langle v | Z_{b_1}(\omega_1) \dots Z_{b_m}(\omega_m), Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1) | v \rangle \\ & := (Z_{b_m}(\omega_m) \dots Z_{b_1}(\omega_1) | v \rangle, Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1) | v \rangle)_0 . \end{aligned}$$

- $\Sigma_{in}^{(n)}$ and $\Sigma_{out}^{(n)}$ have a basis of momentum eigenstates

$$\begin{aligned} & Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1) | v \rangle , & \theta_{\pi(1)} < \dots < \theta_{\pi(n)} , \\ & Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1) | v \rangle , & \theta_{\pi(1)} > \dots > \theta_{\pi(n)} , \end{aligned}$$

respectively, for some fixed permutation $\pi \in S_n$.

It is easy to see that this definition adheres to the usual picture. The factorized scattering theory can equivalently be described in terms of the dual module $\bar{\Sigma}$; the inner products are related by (A.5). This definition does not refer to the Møller operators. We conjecture however that the Møller operators can be constructed in terms of the form factor algebra. Notice that the simple prescription

$$\begin{aligned} \tilde{\Omega}^+ Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1) | v \rangle &= Z_{a_{\pi(n)}}(\theta_{\pi(n)}) \dots Z_{a_{\pi(1)}}(\theta_{\pi(1)}) | v \rangle , \\ \tilde{\Omega}^- Z_{a_n}(\theta_n) \dots Z_{a_1}(\theta_1) | v \rangle &= Z_{a_{\pi(1)}}(\theta_{\pi(1)}) \dots Z_{a_{\pi(n)}}(\theta_{\pi(n)}) | v \rangle , \end{aligned} \quad (\text{A.6})$$

for $\theta_{\pi(1)} < \dots < \theta_{\pi(n)}$ and $\mathcal{S} = \tilde{\Omega}^+(\tilde{\Omega}^-)^{-1}$ reproduces the correct S -matrix elements.

We call a factorised scattering theory *diagonal* if the 2-particle S -matrix is diagonal $S_{ab}^{cd}(\theta) = S_{ab}(\theta) \delta_a^c \delta_b^d$. The charge conjugation matrix becomes $C_{ab} = \delta_{\bar{a}\bar{b}}$, where $a \rightarrow \bar{a}$ is an involution of $\{1, \dots, \dim V\}$. Hermitian unitarity and the crossing invariance eqn. reduce to $S_{ab}(\theta) = S_{ba}(\theta) = S_{ab}(-\theta)^{-1} = S_{ab}^*(-\theta^*)$ and $S_{ab}(i\pi - \theta) = S_{\bar{a}\bar{b}}(\theta)$. For a diagonal scattering theory the additional relation due to the presence of bound states takes a particularly simple form

$$S_{da}(\theta + i\eta(a)) S_{db}(\theta + i\eta(b)) S_{dc}(\theta + i\eta(c)) = 1 , \quad (\text{A.7})$$

where the triplet $(\eta(a), \eta(b), \eta(c))$ is related to the conventional fusing angles[29].

A.2 The extended ZF-algebra

Consider the ZF-algebra (A.1) supplemented by generators $T^\pm(\theta)_a^b$ subject to the relations

$$\begin{aligned} T^\pm(\theta_0)_a^b Z_{a_1}(\theta_1) &= S_{aa_1}^{db_1}(\theta_{01}) Z_{b_1}(\theta_1) T^\pm(\theta_0)_d^b, \\ T^\pm(\theta_0)_a^b \overline{Z}^{a_1}(\theta_1) &= S_{ab_1}^{da_1}(\theta_{10} + 2\pi i) \overline{Z}^{b_1}(\theta_1) T^\pm(\theta_0)_d^b, \end{aligned} \quad (\text{A.8})$$

(and no others). Denote this algebra by $\widetilde{TZ}(S)$. There are several motivations for these exchange relations. First, if one thinks of $T^\pm(\theta)_a^b$ as the generators of a quantum double in its multiplicative presentation, the relations (A8) are characteristic for intertwining operators between quantum double modules. Equivalently, the relations (A8) are designed such that the difference of the left- and the right hand sides of the ZF-algebra generate a tensorial set of ideals in the associative algebra with generators $T^\pm(\theta)_a^b$ and $Z_a(\theta)$, $\overline{Z}^a(\theta)$, subject only to the relations (A.8). A related fact is that the relations (A.8) are essentially the only ones for which the commutator $[T^\pm(\theta)_a^b, \cdot]$ acts as a derivation on products of W -generators. Finally, combined with (T2), Eqn. (A.8) leads to the expression for the adjoint action (2.20).

In the following we will consider the structure of the algebra $\widetilde{TZ}(S)$ in more detail. The one-parameter family of antilinear anti-involutions σ_β of the ZF-algebra can be extended to a two-parameter family on $\widetilde{TZ}(S)$. Ignoring a trivial overall phase it is given by

$$\sigma_{\alpha,\beta}(T^\pm)_a^b(\theta) = \cos \alpha \, T^\pm(\theta^* + i\pi - i\beta)_a^b + i \sin \alpha \, T^\mp(\theta^* + i\pi - i\beta)_a^b, \quad (\text{A.9a})$$

$$\sigma_{\alpha,\beta}(Z_a)(\theta) = C_{aa'} \overline{Z}^{a'}(\theta^* - i\beta), \quad \alpha, \beta \in \mathbb{R}, \quad \sigma_{\alpha,\beta}^2 = id. \quad (\text{A.9b})$$

To verify this first apply $\sigma = \sigma_{\alpha,\beta}$ to the (reversed) relations (A.8), using only (A.9b). This gives

$$\begin{aligned} \sigma(T^\pm)_a^b(\theta_0^*) Z_{a_1}(\theta_1) &= S_{a_1 a}^{b_1 d}(\theta_{01} + i\pi - i\beta) Z_{b_1}(\theta_1) \sigma(T^\pm)_d^b(\theta_0^*), \\ \sigma(T^\pm)_a^b(\theta_0^*) \overline{Z}^{a_1}(\theta_1) &= S_{b_1 a}^{a_1 d}(\theta_{10} + i\pi + i\beta) \overline{Z}^{b_1}(\theta_1) \sigma(T^\pm)_d^b(\theta_0^*), \end{aligned}$$

Consistency with (A.8) requires that $\sigma(T^\pm)_a^b(\theta^*)$ is a linear combination of $T^\pm(\theta_0 + i\pi - i\beta)$ and $T^\mp(\theta_0 + i\pi - i\beta)$. If a and b denote the parameters of the linear combination, the condition $\sigma^2 = id$ implies $|a|^2 + |b|^2 = 1$ and $a^*b + ab^* = 0$. Ignoring a trivial overall phase yields (A.9a).

A.3 Ideals of the extended ZF-algebra

It turns out that $\widehat{TZ}(S)$ contains non-trivial ideals, which can be factored out. Most obvious are the linear ideals. The generators $T^+(\theta)_a^b$ and $T^-(\theta)_a^b$ have by definition identical exchange relations with $Z(S)$ and hence, unless discriminated otherwise, could be identified by factoring out the twosided ideal $T^+(\theta)_a^b - c T^-(\theta)_a^b$. Consistency with $T(S)$ fixes the constant c to be ± 1 . (In fact, in extending the involution σ from $Z(S)$ to $\widehat{TZ}(S)$ we already made use of such a procedure. For any antilinear anti-involution $\sigma T^\pm(\theta)_a^b$ were observed to satisfy the same exchange relations with $Z(S)$ as some linear combination of $T^\pm(\theta + i\pi - i\beta)_a^b$, so that $\sigma T^\pm(\theta)_a^b$ could be identified with a suitable linear combination thereof.) Before turning to the ideals quadratic in $T^\pm(\theta)_a^b$, consider ideals linear in the generators of $Z(S)$ and linear in $T^\pm(\theta)_a^b$. Set

$$(S^{\pm\pm})_a(\theta) = Z_m(\theta) sT^\pm(\theta)_a^m - sT^\pm(\theta)_a^m Z_m(\theta + 2\pi i) , \quad (\text{A.10a})$$

$$(S^{\pm\mp})_a(\theta) = Z_m(\theta) sT^\pm(\theta)_a^m - sT^\mp(\theta)_a^m Z_m(\theta + 2\pi i) , \quad (\text{A.10b})$$

$$(\overline{S}^{\pm\pm})^a(\theta) = \overline{Z}^n(\theta) T^\pm(\theta + 2\pi i)_n^a - T^\pm(\theta + 2\pi i)_n^a \overline{Z}^n(\theta + 2\pi i) , \quad (\text{A.10c})$$

$$(\overline{S}^{\pm\mp})^a(\theta) = \overline{Z}^n(\theta) T^\pm(\theta + 2\pi i)_n^a - T^\mp(\theta + 2\pi i)_n^a \overline{Z}^n(\theta + 2\pi i) . \quad (\text{A.10d})$$

From (A.1) and (A.8) one finds

$$S_a(\theta_1) Z_b(\theta_2) = Z_b(\theta_2) S_a(\theta_1) , \quad \overline{S}^a(\theta_1) \overline{Z}^b(\theta_2) = \overline{Z}^b(\theta_2) \overline{S}^a(\theta_1) , \quad (\text{A.11a})$$

$$S_a(\theta_1) \overline{Z}^b(\theta_2) = \overline{Z}^b(\theta_2) S_a(\theta_1) , \quad \overline{S}^a(\theta_1) Z_b(\theta_2) = Z_b(\theta_2) \overline{S}^a(\theta_1) , \quad (\text{A.11b})$$

where $S_a(\theta)$ stands for one of the operators in (A.10a,b) and $\overline{S}^a(\theta)$ for one of (A.10c,d). In (A.11b) we also assumed $\theta_1 \neq \theta_2$, so that extra terms proportional to $\delta(\theta_{12})$ are absent. Similarly one obtains

$$\begin{aligned} T^\pm(\theta_0)_a^b S_c(\theta_1) &= S_{nc}^{bm}(\theta_{01}) S_m(\theta_1) T^\pm(\theta_0)_a^n + (TTS) , \\ T^\pm(\theta_0)_a^b \overline{S}^c(\theta_1) &= S_{mn}^{cb}(\theta_{10} + 2\pi i) \overline{S}^m(\theta_1) T^\pm(\theta_0)_a^n + (TTS) , \end{aligned} \quad (\text{A.12})$$

where the symbolic notation (TTS) denotes terms linear in the generators of $Z(S)$ and the ideals (A.13) given below. From (A.11), (A.12) one concludes that for generic rapidities all of the operators $S_a(\theta)$, $\overline{S}^a(\theta)$ generate ideals in the quotient algebra obtained from $\widehat{TZ}(S)$ by dividing out the ideals (A.13) but not in $\widehat{TZ}(S)$ itself.

Warning: Observe that the relations (A.8) can be rewritten as

$$sT^\pm(\theta_0)_a^b Z_{a_1}(\theta_1) = S_{a_1 d}^{b_1 b}(\theta_{10}) Z_{b_1}(\theta_1) sT^\pm(\theta_0)_a^d ,$$

By contraction this seems to imply

$$\begin{aligned} Z_m(\theta) sT^+(\theta)_a^m &= -sT^-(\theta)_a^m Z_m(\theta) , \\ Z_m(\theta + 2\pi i) sT^+(\theta)_a^m &= -sT^-(\theta)_a^m Z_m(\theta + 2\pi i) . \end{aligned}$$

using $S_{na}^{bn}(2\pi i) = -\delta_a^b = S_{na}^{bn}(0)$. Similarly, contracting the second relation (A.8) seems to imply

$$\begin{aligned} \overline{Z}^n(\theta) T^\pm(\theta)_n^a &= -T^\pm(\theta)_n^a \overline{Z}^n(\theta) , \\ \overline{Z}^n(\theta + 2\pi i) T^\pm(\theta)_n^a &= -T^\mp(\theta)_n^a \overline{Z}^n(\theta + 2\pi i) . \end{aligned}$$

However, neither of these contracted expressions are valid identities in $\widetilde{TZ}(S)$ or in one of its quotient algebras. The difference of the left- and the right hand sides do not generate ideals in $\widetilde{TZ}(S)$ or in one of its quotient algebras. The reason is that the contracted relations are no longer compatible with the ZF-algebra. If one views the ZF-algebra as arising from dividing out an ideal in the associative algebra generated by $T^\pm(\theta)_a^b$ and $Z_a(\theta)$, $\overline{Z}^a(\theta)$, subject only to the relations (A.8), this incompatibility can be traced back to the following fact: Because of mixing effects, an invariant subset of a tensorial set of ideals may, but need not, generate an ideal by itself. In the case at hand e.g. $T^\pm(\theta_0)_e^f [Z_a(\theta_1) Z_b(\theta_2) - S_{ab}^{dc}(\theta_{12}) Z_c(\theta_2) Z_d(\theta_1)]$ generates a tensorial set of ideals, its contraction on the $e = a$ index does not. The above result on the operators $S_a(\theta)$ and $\overline{S}^a(\theta)$ shows that nevertheless some ‘deformed’ version of the contracted relations (A.8) can consistently be imposed in a suitable quotient algebra of $\widetilde{TZ}(S)$.

Consider now the ideals quadratic in $T^\pm(\theta)_a^b$. We claim that each of the tensorial sets

$$S_{ab}^{mn}(\theta_{12}) T^\pm(\theta_2)_n^d T^\pm(\theta_1)_m^c - T^\pm(\theta_1)_a^m T^\pm(\theta_2)_b^n S_{mn}^{cd}(\theta_{12}) \quad (\text{A.13a})$$

$$S_{ab}^{mn}(\theta_{12}) T^\pm(\theta_2)_n^d T^\mp(\theta_1)_m^c - T^\mp(\theta_1)_a^m T^\pm(\theta_2)_b^n S_{mn}^{cd}(\theta_{12}) \quad (\text{A.13b})$$

generates a twosided ideal in $\widetilde{TZ}(S)$. Moreover the ideals (A.13) collectively are invariant under σ . To verify this let $I[T^\pm, T^\pm]$, $I[T^\pm, T^\mp]$ denote the tensorial sets appearing in (A.13a,b), respectively. One can then verify by direct computation that $I[T^\pm, T^\pm] Z(S) = Z(S) I[T^\pm, T^\pm]$, which is claim (A.13a). Since $T^+(\theta)_a^b$ and $T^-(\theta)_a^b$ have identical exchange relations with $Z(S)$ one can also substitute $T^\mp(\theta)_a^b$ for one of the $T^\pm(\theta)_a^b$ pairs in $I[T^\pm, T^\pm]$, which yields (A.13b). For the invariance under σ then note the following fact (*): For any antilinear anti-involution σ of $\widetilde{TZ}(S)$ that preserves $Z(S)$, if $I(T, T)$ generates an ideal, so does $\sigma I(T, T) \simeq I(\sigma T^*, \sigma T^*)$, where $T^*(\theta)_a^b := T(\theta^*)_a^b$. Here we use \simeq to indicate equality modulo a relabeling of rapidities and/or raising and lowering of indices by means of the charge conjugation matrix. The fact (*) then follows from the property (2.3) of

the S -matrix. Applied to the ideal $I[T^\pm, T^\pm]$ and the antilinear anti-involution (A.9) one concludes that also

$$\begin{aligned} \sigma I[T^\pm, T^\pm] &\simeq \cos^2 \alpha I[T^\pm, T^\pm] + i \cos \alpha \sin \alpha (I[T^\pm, T^\mp] + I[T^\mp, T^\pm]) \\ &\quad - \sin^2 \alpha I[T^\mp, T^\mp] \end{aligned}$$

generates an ideal and similarly for $I[T^\pm, T^\mp]$. As a consistency check note that the traces of the ideals (A.13) (after contracting with S^{-1}) generate the expected scalar ideals.

There are many more quadratic ideals in $\widetilde{TZ}(S)$. Set

$$\begin{aligned} F^{\pm\pm}(\theta)_a^b &:= C_{aa'} C^{mn} T^\pm(\theta)_m^{a'} T^\pm(\theta + i\pi)_n^b, \\ F^{\mp\pm}(\theta)_a^b &:= C_{aa'} C^{mn} T^\mp(\theta)_m^{a'} T^\pm(\theta + i\pi)_n^b, \end{aligned} \quad (\text{A.14})$$

which is a σ -invariant set of operators. One checks that even each of the components $F^{\pm\pm}(\theta)_a^b$ and $F^{\mp\pm}(\theta)_a^b$ separately generates an ideal i.e.

$$F(\theta)_a^b Z(S) = Z(S) F(\theta)_a^b,$$

where $F(\theta)_a^b$ stands for any of the operators (A.14). In particular the $F(\theta)_a^b$'s collectively can consistently be taken to define a tensorial set of ideals, in accordance with the index structure. Notice also that $F(\theta)_a^b - \lambda \delta_a^b$ again generate tensorial ideals for any $\lambda \in \mathbb{C}$. For a product of T -generators with relative rapidities equal to $+i\pi$ the situation is more subtle. Define

$$\begin{aligned} G^{\pm\pm}(\theta_1, \theta_2)_a^b &:= C^{bb'} C_{mn} T^\pm(\theta_1 + i\pi)_a^m T^\pm(\theta_2)_{b'}^n, \\ G^{\mp\pm}(\theta_1, \theta_2)_a^b &:= C^{bb'} C_{mn} T^\mp(\theta_1 + i\pi)_a^m T^\pm(\theta_2)_{b'}^n, \end{aligned} \quad (\text{A.15})$$

which is a σ -invariant set of operators. Each of them satisfies the relations

$$\begin{aligned} G(\theta_1, \theta_2)_a^b Z_{a_3}(\theta_3) &= S_{a_3d}^{eb}(\theta_{32} + \pi i) S_{ae}^{cb_3}(\theta_{13} + i\pi) Z_{b_3}(\theta_3) G(\theta_1, \theta_2)_c^d, \\ G(\theta_1, \theta_2)_a^b \overline{Z}^{a_3}(\theta_3) &= S_{ed}^{a_3b}(\theta_{23} - \pi i) S_{ab_3}^{ce}(\theta_{31} + \pi i) \overline{Z}^{b_3}(\theta_3) G(\theta_1, \theta_2)_c^d. \end{aligned} \quad (\text{A.16})$$

If $G(\theta_1, \theta_2)_a^b$ however were to generate a tensor ideal, so should $G(\theta_1, \theta_2)_a^b - \lambda \delta_a^b$ for $\lambda \neq 0$. The latter requires

$$\begin{aligned} S_{a_3d}^{eb}(\theta_{32} + i\pi) S_{ae}^{cb_3}(\theta_{13} + i\pi) \delta_c^d &= \delta_a^b \delta_{a_3}^{b_3}, \\ S_{ed}^{a_3b}(\theta_{23} - i\pi) S_{ab_3}^{ce}(\theta_{31} + i\pi) \delta_c^d &= \delta_a^b \delta_{b_3}^{a_3}. \end{aligned}$$

Clearly this will not be the case for generic arguments. By fine-tuning the arguments one finds that

$$G^{\pm\pm}(\theta, \theta)_a^b - \lambda \delta_a^b \mathbb{1}, \quad G^{\mp\pm}(\theta, \theta)_a^b - \lambda \delta_a^b \mathbb{1}, \quad \lambda \in \mathbb{C}, \quad (\text{A.17})$$

generate tensorial ideals. Notice however that the traces $G(\theta, \theta)_a^a$ do not generate scalar ideals by themselves.[†] Scalar ideals are in fact obtained from the traces of $G(\theta_1, \theta_2)_a^b$ for a different fine-tuning of the arguments in (A.15), but these coincide with the traces of $F(\theta)_a^b$

$$F^{\pm\pm}(\theta)_a^a = G^{\pm\pm}(\theta, \theta + 2\pi i)_a^a, \quad F^{\mp\pm}(\theta)_a^a = G^{\mp\pm}(\theta, \theta + 2\pi i)_a^a. \quad (\text{A.18})$$

We expect (but have not proved) that the above provides a complete list of quadratic ideals in $\widetilde{TZ}(S)$. We shall not discuss higher order ideals here. It is not hard to see that bound state poles in the S-matrix will give rise to cubic relations among the $Z(S)$ generators for special, fine-tuned triples of rapidities. This will induce additional consistency relations for the extended ZF-algebra and will affect the structure of its higher order ideals. Clearly this will be sensitive to the details of the bound state structure and has to be discussed for each model separately.

A.4 Some quotient algebras

Having identified the ideals of $\widetilde{TZ}(S)$ one can obtain consistent quotient algebras by dividing out an appropriate subset of the ideals. The question what ideals one decides to divide out, depends on the class of representations of the quotient algebra one is interested in. In particular, in the context of form factors one wishes to keep the generators $T^+(\theta)_a^b$ and $T^-(\theta)_a^b$ distinct, so that one is not allowed to divide out the linear ideal $T^+(\theta)_a^b \pm T^-(\theta)_a^b$ in $\widetilde{TZ}(S)$. Still, one can divide out other ideals and an immediate corollary is the consistency of the algebra $F_*(S)$. To see this, observe that the operators $Z_a(\theta)$ generate a subalgebra of the ZF-algebra. All the results on the structure of the ideals in the extended ZF-algebra of course carry over to this subalgebra, extended in a similar fashion. Doing this, the rapidity variable in $Z_a(\theta)$ initially is real, but for the purely algebraic purposes considered here, it can also be extended to complex values. One can then divide out the ideals (A.13), $F^{\pm\pm}(\theta)_a^b - \delta_a^b$, $G^{\pm\pm}(\theta, \theta)_a^b - \delta_a^b$ and $(S^{+-})_a(\theta)$. The resulting associative algebra is by construction consistent and is isomorphic to $F_*(S)$. Clearly, this consistency is not affected by imposing the residue conditions (R \pm).

As mentioned before, an algebra obtained by dividing out the linear ideal $T^+(\theta)_a^b \pm T^-(\theta)_a^b$ will not be of relevance in the context of form factors. Nevertheless, one can consider the quotient algebra obtained by dividing out the maximal (linear and quadratic) ideal I_{max} in $\widetilde{TZ}(S)$. The resulting algebra may be viewed as the symmetry algebra of a

[†]This is not in conflict with the tensorial character of $G(\theta, \theta)_a^b$. Because of mixing effects an invariant subset of a tensorial set of ideals may, but need not generate an ideal by itself.

factorized scattering and will be denoted by $TZ(S)$, i.e.

$$TZ(S) = \widetilde{TZ}(S) / I_{max} . \quad (\text{A.19})$$

By construction, $TZ(S)$ is a consistent associative algebra. Explicitly, the defining relations are (A.1) for the $Z(S)$ subalgebra and

$$\begin{aligned} S_{12}(\theta_{12}) T_2(\theta_2) T_1(\theta_1) &= T_1(\theta_1) T_2(\theta_2) S_{12}(\theta_{12}) , \\ C^{mn} T_m^a(\theta) T_n^b(\theta + i\pi) &= C^{ab} \mathbb{1} , \quad C_{mn} T_a^m(\theta + i\pi) T_b^n(\theta) = C_{ab} \mathbb{1} , \end{aligned}$$

for the $T(S)$ subalgebra. The mixed relations are (A.8) with $T^\pm(\theta)_a^b$ replaced by $T_a^b(\theta)$ and

$$\begin{aligned} C^{mn} Z_m(\theta) T(\theta + i\pi)_n^a &= C^{mn} T(\theta + i\pi)_n^a Z_m(\theta + 2\pi i) , \\ \overline{Z}^n(\theta) T(\theta + 2\pi i)_n^a &= T(\theta + 2\pi i)_n^a \overline{Z}^n(\theta + 2\pi i) . \end{aligned} \quad (\text{A.20})$$

Except for the last relations the algebra $TZ(S)$ also appeared in [26]. The algebra $TZ(S)$ still is endowed with a linear anti-homomorphism s and an antilinear anti-involution σ_β . Both are obtained in the obvious way from Eqn. (A.10) and (A.9), respectively.

As seen in section A.1, a factorized scattering theory can be described in terms of Fock-type representations of the $Z(S)$ algebra alone. Having enlarged the $Z(S)$ algebra to $TZ(S)$ it is natural to extend the previous representations to Fock-type representations of $TZ(S)$ by requiring the existence of a vector $|v_0\rangle \in \Sigma$ satisfying $\overline{Z}^a(\theta)|v_0\rangle = 0$ and

$$T_a^b(\theta)|v_0\rangle = \delta_a^b|v_0\rangle , \quad \sigma_0(T_a^b(\theta))|v_0\rangle = T_a^b(\theta + i\pi)|v_0\rangle = \delta_a^b|v_0\rangle , \quad \theta \in \mathbb{R} . \quad (\text{A.21})$$

The first Eqn. fixes the action of $T_a^b(\theta)$ on $Z(S)|v_0\rangle$. The explicit form is conveniently obtained from the adjoint action (2.20). The second Eqn. (A.21) guarantees the consistency with the inner product (A.4). From (A.5) one can also work out the description in terms of the dual modules, which is equivalent and independent.

A.5 The real rapidity algebra $R(S)$

All quotient algebras of $\widetilde{TZ}(S)$ contain the ZF-algebra as a subalgebra. In this section we study a modification of such an algebra, where this is no longer the case in that the coefficient of the delta distribution term becomes operator-valued. From $\widetilde{TZ}(S)$ one divides out the ideals implementing the relations of $T(S)$ and $S_a^{\pm\pm}(\theta)$, $(\overline{S}^{\pm\pm})^a(\theta)$. In addition one

modifies the δ -distribution term in the ZF-algebra to become operator-valued and proportional to $[C^{aa'} L_{a'b}^\pm(\theta_2) - \delta_a^b]$. This can be considered as an implementation of Smirnov's residue formula (2.15), with the (crucial) simplification that the pole singularity has been replaced by a delta function singularity. The resulting algebra $R(S)$ is a consistent extension of the ZF-algebra and can be viewed as a reduced version of the alternative form factor algebra $F(S)$ described in appendix B.

We define the algebra $R(S)$ as an associative algebra with generators $Z_a(\theta)$, $\bar{Z}_a(\theta)$, $T^\pm(\theta)_a^b$ for $\theta \in i\pi\mathbb{Z} + \mathbb{R}$, a unit $\mathbb{1}$ and the generators P_μ , $\epsilon_{\mu\nu}K$ of the 1+1 dimensional Poincaré algebra. In a slight abuse of notation we shall still call $R(S)$ a 'real rapidity algebra'. The defining relations are that of $T(S)$ (restricted to rapidities in $i\pi\mathbb{Z} + \mathbb{R}$), together with

$$(TZ)_R \quad \begin{aligned} T^\pm(\theta_0)_a^b Z_{a_1}(\theta_1) &= S_{aa_1}^{db_1}(\theta_{01}) Z_{b_1}(\theta_1) T^\pm(\theta_0)_d^b, \\ T^\pm(\theta_0)_a^b \bar{Z}^{a_1}(\theta_1) &= S_{ab_1}^{da_1}(\theta_{10} + 2\pi i) \bar{Z}^{b_1}(\theta_1) T^\pm(\theta_0)_d^b, \end{aligned}$$

replacing (TW). Further

$$(S)_R \quad \begin{aligned} C^{mn} Z_m(\theta) T^+(\theta + i\pi)_n^a &= C^{mn} T^+(\theta + i\pi)_n^a Z_m(\theta + 2\pi i), \\ \bar{Z}^n(\theta) T^-(\theta + 2\pi i)_n^a &= T^-(\theta + 2\pi i)_n^a \bar{Z}^n(\theta + 2\pi i), \end{aligned}$$

replacing (S). Finally the (WW) relations get replaced by a modified ZF-algebra with operator-valued coefficients of the singular terms

$$(ZZ)_R \quad \begin{aligned} Z_a(\theta_1) Z_b(\theta_2) &= S_{ab}^{dc}(\theta_{12}) Z_c(\theta_2) Z_d(\theta_1) \\ \bar{Z}^a(\theta_1) \bar{Z}^b(\theta_2) &= S_{dc}^{ab}(\theta_{12}) \bar{Z}^c(\theta_2) \bar{Z}^d(\theta_1) \\ \bar{Z}^a(\theta_1) Z_b(\theta_2) &= S_{db}^{ac}(\theta_{21}) Z_c(\theta_2) \bar{Z}^d(\theta_1) + [C^{aa'} L_{a'b}^+(\theta_2) - \delta_a^b] 2\pi\delta(\theta_{12}) \\ Z_a(\theta_1) \bar{Z}^b(\theta_2) &= S_{ac}^{db}(\theta_{21} + 2\pi i) \bar{Z}^c(\theta_2) Z_d(\theta_1) + [C^{bb'} L_{ab'}^-(\theta_1) - \delta_a^b] 2\pi\delta(\theta_{12}), \end{aligned}$$

where all rapidities are real modulo $i\pi$ and $L_{ab}^\pm(\theta)$ are defined as in (2.11). The delta functions are to be read as $\delta(\theta) := \delta(\text{Re } \theta) \delta_{Im \theta, 0}$, where the Kronecker delta is $2\pi i$ -periodic.

The proof that $R(S)$ is a consistently defined associative algebra does not follow from the previous results. Because of the operator-valued coefficients of the delta function, $R(S)$ is not built from subalgebras generated by $T^\pm(\theta)_a^b$ and $Z_a(\theta)$, $\bar{Z}^a(\theta)$, respectively. This induces some new features in demonstration of consistency. It is convenient to split the discussion into the following items.

- (a) Iteration of $(ZZ)_R$ and associativity of $(ZZ)_R$.

(b) Consistency of $(TZ)_R$ with $(ZZ)_R$

(c) Consistency of $(S)_R$ with $(ZZ)_R$

(a) Iteration of $(ZZ)_R$: Applying the $(ZZ)_R$ relations to the product of Z -generators on the r.h.s. should reproduce the l.h.s. For the first two relations in $(ZZ)_R$ this is trivial, and for the last two it is a consequence of equation (2.12). Associativity of $(ZZ)_R$: To work out the conditions imposed by associativity of the multiplication in triple products of ZF-generators one rearranges both sides of the identity $(V_1 V_2) V_3 = V_1 (V_2 V_3)$ by repeated use of $(ZZ)_R$ until the order of the factors is reversed. Here $V_a(\theta)$ stands for either $Z_a(\theta)$ or $\bar{Z}^a(\theta)$. The terms cubic in V on both sides are found to coincide by means of the Yang Baxter equation. For triple products where both $Z_a(\theta)$ and $\bar{Z}^a(\theta)$ enter there will three additional terms linear in V on either side. Their matching conditions are found to be equivalent to

$$\begin{aligned}
L_{ad}^+(\theta_0) Z_{a_1}(\theta_1) S_{bc}^{a_1 d}(\theta_{10}) &= S_{ab}^{db_1}(\theta_{01} + i\pi) Z_{b_1}(\theta_1) L_{dc}^+(\theta_0) , \\
L_{ad}^+(\theta_0) \bar{Z}^{a_1}(\theta_1) S_{a_1 c}^{bd}(\theta_{01}) &= S_{ab_1}^{db}(\theta_{10} + i\pi) \bar{Z}^{b_1}(\theta_1) L_{dc}^+(\theta_0) , \\
L_{ad}^-(\theta_0) Z_{a_1}(\theta_1) S_{bc}^{a_1 d}(\theta_{10} - i\pi) &= S_{ab}^{db_1}(\theta_{01}) Z_{b_1}(\theta_1) L_{dc}^-(\theta_0) , \\
L_{ad}^-(\theta_0) \bar{Z}^{a_1}(\theta_1) S_{a_1 c}^{bd}(\theta_{01} - i\pi) &= S_{ab_1}^{db}(\theta_{10}) \bar{Z}^{b_1}(\theta_1) L_{dc}^-(\theta_0) .
\end{aligned} \tag{A.22}$$

All of them can be checked to be identities in $R(S)$ and to be compatible with (2.12).

(b) Consistency of $(TZ)_R$ with $(ZZ)_R$: Further consistency conditions arise if one pushes $T^\pm(\theta_0)_a^b$ through both sides of the equation $(TZ)_R$. In technical terms the relations $(ZZ)_R$ have to correspond to a twosided ideal in the algebra generated by $T(S)$ and the $(TZ)_R$ relations alone. Explicitly one finds

$$\begin{aligned}
S_{ab}^{mn}(\theta_{01}) L_{cn}^+(\theta_1) T^\pm(\theta_0)_m^d &= S_{ca}^{mn}(\theta_{10} + i\pi) T^\pm(\theta_0)_n^d L_{mb}^+(\theta_1) , \\
S_{ab}^{mn}(\theta_{01} + i\pi) L_{cn}^-(\theta_1) T^\pm(\theta_0)_m^d &= S_{ca}^{mn}(\theta_{10}) T^\pm(\theta_0)_n^d L_{mb}^-(\theta_1) ,
\end{aligned} \tag{A.23}$$

together with the same equations where $L_{ab}^\pm(\theta)$ is replaced by C_{ab} . Again all of them can be checked to be identities in $R(S)$.

(c) Consistency of $(S)_R$ with $(ZZ)_R$. The claim here is that if one defines

$$\begin{aligned}
s(Z_a)(\theta) &:= Z_m(\theta) sT^+(\theta)_a^m = sT^+(\theta)_a^m Z_m(\theta + 2\pi i) , \\
s(\bar{Z}^a)(\theta) &:= \bar{Z}^n(\theta) T^-(\theta + 2\pi i)_n^a = T^-(\theta + 2\pi i)_n^a \bar{Z}^n(\theta + 2\pi i) ,
\end{aligned} \tag{A.24}$$

both of the expressions on the r.h.s have the same exchange relations with the ZF-operators. This is indeed the case. For example one finds consistently

$$s(Z_a)(\theta_1) Z_b(\theta_2) = Z_b(\theta_2) s(Z_a)(\theta_1) ,$$

$$s(Z_a)(\theta_1)\overline{Z}^b(\theta_2) = \overline{Z}^b(\theta_2)s(Z_a)(\theta_1) + [sT^+(\theta_1)_a^b - sT^-(\theta_1)_a^b] 2\pi\delta(\theta_{12}) , \quad (\text{A.25})$$

together with two similar eqn.s where the roles of Z and \overline{Z} are interchanged. This concludes the demonstration of the consistency of $R(S)$.

The last point (c) also allows one to extend the antipode map s on $T(S)$ to a linear anti-homomorphism s on $R(S)$ by taking (A.24) as its action on the Z, \overline{Z} -generators. One verifies that s indeed acts as a linear anti-homomorphism on the $(TZ)_R, (S)_R$ and $(ZZ)_R$ relations. From (A.25) one also sees that for $(\theta_{12} \bmod 2\pi i) \neq 0$ the original Z, \overline{Z} generators commute with the s transformed ones. Consistent with that one has

$$s(L_{ab}^\pm)(\theta_1) Z_c(\theta_2) = Z_c(\theta_2) s(L_{ab}^\pm)(\theta_1) , \quad s(L_{ab}^\pm)(\theta_1) \overline{Z}^c(\theta_2) = \overline{Z}^c(\theta_2) s(L_{ab}^\pm)(\theta_1) ,$$

valid for generic rapidities. Finally note that the algebra $TZ(S)$ described in section A.4 can be recovered from $R(S)$ by means of the reduction

$$T^+(\theta)_a^b = -T^-(\theta)_a^b =: T_a^b(\theta) .$$

As in the case of the ZF-algebra one can formally consider $R(S)$ also for complex rapidities. Let $R(S)_C$ denote the ‘complexified’ algebra $R(S)$. We claim that $R(S)_C$ is endowed with a one parameter family of antilinear anti-involution σ_β given by

$$\begin{aligned} \sigma_\beta(T^\pm)_a^b(\theta) &= T^\mp(\theta^* + i\pi - i\beta)_a^b , \\ \sigma_\beta(Z_a)(\theta) &= C_{aa'} \overline{Z}^{a'}(\theta^* - i\beta) , \quad \beta \in \mathbb{R} , \quad \sigma_\beta^2 = id . \end{aligned} \quad (\text{A.26})$$

From section A.2 it follows that σ_β is an antilinear anti-involution of the exchange relations $(TZ)_R$. Further $Z_a(\theta) \rightarrow \sigma_\beta(Z_a)(\theta)$ is an antilinear anti-involution of the ZF-algebra (where the coefficients of the singular terms are constant). Having modified the ZF-algebra as in $(ZZ)_R$, the invariance of the last two Eqn.s has to be re-examined. The definitions imply

$$\sigma_\beta(L_{ab}^\pm)(\theta) = L_{ba}^\pm(\theta^* \mp i\beta) ,$$

from which it is easy to check that σ_β acts as an antilinear anti-involution also on the last two eqn.s $(ZZ)_R$. It remains to verify the consistency with the relations $(S)_R$. Indeed one checks that

$$\sigma_\beta(T^+)(\theta + i\pi)_n^a \sigma_\beta(Z_m)(\theta) C^{mn} = \sigma_\beta(Z_m)(\theta + 2\pi i) \sigma_\beta(T^+)(\theta + i\pi)_n^a C^{mn}$$

holds by means of the second equation $(S)_R$ and vice versa. Together it follows that (A.26) indeed defines a one parameter family of antilinear anti-involutions of $R(S)_C$. Return now

to $(ZZ)_R$. In terms of $\sigma_\beta(Z_a)(\theta)$ the last two relations can be rewritten as

$$\begin{aligned}
\sigma_\beta(Z_a)(\theta_1^*) Z_b(\theta_2) &= S_{ba}^{cd}(\theta_{12} + i\pi - i\beta) Z_c(\theta_2) \sigma_\beta(Z_d)(\theta_1^*) \\
&+ [L_{ab}^+(\theta_2) - C_{ab}] 2\pi\delta(\theta_{12} - i\beta) , \\
Z_a(\theta_1) \sigma_\beta(Z_b)(\theta_2^*) &= S_{ab}^{dc}(\theta_{12} - i\pi + i\beta) \sigma_\beta(Z_c)(\theta_2^*) Z_d(\theta_1) \\
&+ [L_{ab}^-(\theta_1) - C_{ab}] 2\pi\delta(\theta_{12} + i\beta) .
\end{aligned} \tag{A.27}$$

Comparing now (WW) with (A.27) specialized to $\beta = \pi$, this suggests that one can actually combine both of the generators $Z_a(\theta)$ and $\overline{Z}^a(\theta)$ into a single operator $W_a(\theta)$. Define

$$\begin{aligned}
W_a(\theta - i\epsilon) &= \begin{cases} Z_a(\theta) , & \theta \in \mathbb{R} \\ C_{aa'} \overline{Z}^{a'}(\theta - i\pi) , & \theta \in \mathbb{R} + i\pi . \end{cases} \\
W_a(\theta + i\epsilon) &= \begin{cases} C_{aa'} \overline{Z}^{a'}(\theta - i\pi) , & \theta \in \mathbb{R} \\ Z_a(\theta) , & \theta \in \mathbb{R} - i\pi , \end{cases}
\end{aligned} \tag{A.28}$$

where the limit $\epsilon \rightarrow 0^+$ is to be taken. One finds that all of the relations $(ZZ)_R$ translate into

$$W_a(\theta_1) W_b(\theta_2) = S_{ab}^{dc}(\theta_{12}) W_c(\theta_2) W_d(\theta_1) + [L_{ab}^\pm(\theta_2) - C_{ab}] 2\pi\delta(\theta_{12} \mp i\pi) , \tag{A.29}$$

where $L_{ab}^\pm(\theta)$ are the same as in (2.11) and the upper/lower case options correspond to $\pm \text{Im } \theta > 0$, respectively. Similarly both of the relations $(TZ)_R$ translate into (TW) and $(S)_R$ translates into

$$C^{mn} W_m(\theta) T^\pm(\theta + i\pi)_n^a = C^{mn} T^\pm(\theta + i\pi)_n^a W_m(\theta + 2\pi i) , \tag{A.30}$$

where the upper case corresponds to $\theta \in \mathbb{R} + i\epsilon, \mathbb{R} + i\epsilon + i\pi$ and the lower case to $\theta \in \mathbb{R} - i\epsilon, \mathbb{R} - i\epsilon - i\pi$.

In summary, by means of the correspondence (A.28), the algebra $R(S)$ is isomorphic to an associative algebra with generators $W_a(\theta)$, $T^\pm(\theta)_a^b$, subject to the relations of $T(S)$ together with (TW), (A.29) and (A.30). The W -generators initially are defined for rapidities in an $i\epsilon$ -neighbourhood of the axis $\mathbb{R} + i\pi\mathbb{Z}$, but it is easy to see that the algebraic consistency is preserved if the range of definition is extended to generic complex rapidities. We shall denote the resulting algebra by $F_R(S)$. The antilinear anti-involution (A.26) on $R(S)$ induces an antilinear anti-involution on $F_R(S)$ of the same form as (2.8). For $\theta_{12} \neq \pm i\pi$, the extra term on the r.h.s of (A.29) is absent. One can then also replace (A.30) by (S), so that for $\theta_{12} \neq \pm i\pi$ $F_R(S)$ is isomorphic to $F_*(S)$. For relative

rapidities that are purely imaginary, the extra term in (A.29) is reminiscent of Smirnov's residue formula (2.15), except for the (crucial) difference that the pole singularity has been replaced by a delta function singularity. It is easy to see that the naive device to replace $\delta(\theta)$ by $1/\theta$ or $1/(\theta \pm i0)$ would *not* be algebraically consistent. The reason is that for $\theta_{12} \neq \pm i\pi$, a product of W -generators has different exchange relations with $T^\pm(\theta)_a^b$ as $L_{ab}^\pm(\theta_2)/(\theta_{12} \pm i\pi)$. The algebraic consistency off the singularity $\theta_{12} = \pm i\pi$ can be restored by introducing an extra generator $C_{ab}(\theta_1, \theta_2)$ and leads to an alternative form factor algebra.

B. Alternative form factor algebra

From a technical viewpoint, the main problem in constructing an algebra that implements the form factor axioms is to reconcile the (WW) exchange relations with a residue formula of the type (2.15). One would like to implement (2.15) in terms of a simpler, numerical residue condition. One way to achieve this was described in section 2 and lead to the form factor doublet $F_\pm(S)$. Here we present an alternative extension of the algebra $F_*(S)$, which may be viewed as an extension of the algebra $F_R(S)$ 'off the singularity'. The advantage is that the form factors directly arise from T -invariant functionals over $F(S)$ (instead of a sum of two such functionals as for $F_\pm(S)$). Further, the algebras $F_R(S)$ or $R(S)$ can directly be recovered by means of a reduction process. An important disadvantage of $F(S)$ is that the additional generator $C_{ab}(\theta_1, \theta_2)$ is difficult to construct in a realization.

B.1 Definition of the algebra $F(S)$

Recall that in $F_*(S)$ the product of W generators is defined only when all relative rapidities have a nonvanishing real part. In order to extend the product to cases where one of the relative rapidities is purely imaginary, a further generator $C_{ab}(\theta_1, \theta_2)$, $\theta_1, \theta_2 \in \mathbb{C}$ is needed. The form factor algebra $F(S)$ is defined to be the associative extension of the algebra $F_*(S)$ by the generator $C_{ab}(\theta_1, \theta_2)$, subject to the following relations:

$$(E) \quad W_a(\theta_1) W_b(\theta_2) = \begin{cases} E_{ab}^+(\theta_1, \theta_2), & \text{Re } \theta_{12} \rightarrow 0^+, \quad 2\pi > \text{Im } \theta_{12} > 0 \\ E_{ab}^-(\theta_1, \theta_2), & \text{Re } \theta_{12} \rightarrow 0^-, \quad -2\pi < \text{Im } \theta_{12} < 0, \end{cases}$$

where

$$\begin{aligned} E_{ab}^+(\theta_1, \theta_2) &:= T^-(\theta_1)_a^m [T^+(\theta_2)_b^n - T^-(\theta_2)_b^n] C_{nm}(\theta_2, \theta_1), \\ E_{ab}^-(\theta_1, \theta_2) &:= [T^+(\theta_1)_a^m - T^-(\theta_1)_a^m] T^+(\theta_2)_b^n C_{nm}(\theta_2, \theta_1). \end{aligned} \quad (B.1)$$

The operator $C_{ab}(\theta_1, \theta_2)$ describes the singularity structure in products of W generators. It is defined to have the following properties:

$$(C1) \quad C_{ab}(\theta_1, \theta_2) W_c(\theta_3) = W_c(\theta_3) C_{ab}(\theta_1, \theta_2) .$$

Further

$$(C2) \quad C_{ab}(\theta_1, \theta_2) = S_{ab}^{dc}(\theta_{21}) C_{cd}(\theta_2, \theta_1) , \quad \theta_{12} \neq \pm i\pi ,$$

where relative rapidities $\pm i\pi$ have to be excluded to ensure the consistency of (C3) and (C5) below.

$$(C3) \quad S_{nb}^{dm}(\theta_{10}) T^\pm(\theta_0)_a^n C_{mc}(\theta_1, \theta_2) = S_{nc}^{dm}(\theta_{02} + 2\pi i) C_{bm}(\theta_1, \theta_2) T^\pm(\theta_0)_a^n , \\ \theta_{10} \neq 0, 2\pi i, \quad \theta_{20} \neq 0, 2\pi i ,$$

At the excluded points the contracted relation (C3) gets modified to

$$(C4) \quad T^-(\theta_1)_a^m C_{mb}(\theta_1, \theta_2) = C_{bm}(\theta_2, \theta_1 + 2\pi i) T^+(\theta_1)_a^m ,$$

for generic rapidities θ_1, θ_2 . Finally we require that for fixed θ_1 the operator $C_{ab}(\theta_1, \theta_2)$ is meromorphic in θ_2 with a simple pole at $\theta_2 = \theta_1 + i\pi$; the residue being given by

$$(C5) \quad 2\pi i \text{res} C_{ab}(\theta, \theta + i\pi) = C_{ab} ,$$

which is consistent with (C3).

Implicit in this definition, of course, is the presupposition that by means of (C1) – (C4), the prescription (E) is consistent with the relations of the algebra $F_*(S)$. The consistency of (E) with $F_*(S)$ imposes the following conditions on $E_{ab}^\pm(\theta_1, \theta_2)$

$$E_{ab}^\pm(\theta_1, \theta_2) = S_{ab}^{dc}(\theta_{12}) E_{cd}^\mp(\theta_2, \theta_1) , \quad \text{Re } \theta_{12} \neq 0 , \quad (B.2)$$

$$S_{ab}^{mn}(\theta_{02}) E_{cn}^+(\theta_1, \theta_2) W_m(\theta_0) = S_{ca}^{mn}(\theta_{10}) W_n(\theta_0) E_{mb}^+(\theta_1, \theta_2) , \quad (B.3)$$

$$S_{ab}^{mn}(\theta_{02}) E_{cn}^+(\theta_1, \theta_2) T^\pm(\theta_0)_m^d = S_{ca}^{mn}(\theta_{10}) T^\pm(\theta_0)_n^d E_{mb}^+(\theta_1, \theta_2) , \quad (B.4)$$

$$sT^-(\theta_1)_a^m E_{mb}^+(\theta_1 + 2\pi i, \theta_2) = E_{bm}^-(\theta_2, \theta_1) sT^+(\theta_1)_a^m . \quad (B.5)$$

In Eqn.s (B.3), (B.4) one can also replace E_{ab}^+ by E_{ab}^- , the replacement being consistent with (B.2). The origin of (B.2) – (B.4) is obvious. The last condition is needed to ensure the consistency of (S) and (E) or equivalently of (2.7) and (E). Eqn. (B.5) arises from rearranging $W_a(\theta_1 + 2\pi i) W_b(\theta_2) = E_{ab}^+(\theta_1 + 2\pi i, \theta_2)$ and $W_a(\theta_1) W_b(\theta_2 + 2\pi i) = E_{ab}^-(\theta_1, \theta_2 + 2\pi i)$ for $\pm Im \theta_{12} > 0$, respectively. Notice that one can only shift the first, respectively second, argument in $W_a(\theta_1) W_b(\theta_2)$ by $2\pi i$ without violating the condition $\pm Im \theta_{12} > 0$. As a consequence, the reversed Eqn.s $sT^-(\theta)_a^m E_{mb}^-(\theta_1 + 2\pi i, \theta_2) = E_{bm}^+(\theta_2, \theta_1) sT^+(\theta_1)_a^m$

do not arise as consistency conditions. It will become clear below that (B2) – (B5) are in fact a complete set of consistency conditions. One can now check from the definitions (B1) that these Eqn.s hold by means of (C1) – (C4). Of course, one still has to check that (C1) – (C4) are consistent with $F_*(S)$. Explicitely, this means that the the consistency conditions arising from "pushing the $T^\pm(\theta)_a^b$ or the $W_a(\theta)$ generators through the relations (C1)–(C4)" all are identities within $F(S)$. For the W -generators this is trivial due to (C1). For the $T(S)$ generators it is easy to see that it suffices to push $T^\pm(\theta)_a^b$ through the relations (C3) and (C4). Let us illustrate the procedure for (C4). For the l.h.s one finds

$$T^\pm(\theta_3)_c^d T^+(\theta_1)_a^m C_{mb}(\theta_1, \theta_2) = S_{ca}^{kl}(\theta_{31}) S_{eb}^{dq}(\theta_{32} + 2\pi i) T^+(\theta_1)_l^p C_{pq}(\theta_1, \theta_2) T^\pm(\theta_3)_k^e ,$$

using (T1) and (C3), while the r.h.s. gives

$$\begin{aligned} & T^\pm(\theta_3)_c^d C_{bm}(\theta_2, \theta_1 + 2\pi i) T^-(\theta_1)_a^m \\ &= S_{ca}^{kl}(\theta_{31}) S_{eb}^{dq}(\theta_{32} + 2\pi i) C_{qp}(\theta_2, \theta_1 + 2\pi i) T^-(\theta_1)_l^p T^\pm(\theta_3)_k^e . \end{aligned}$$

Using now (C4) again on the r.h.s one obtains an identity. The consistency of (C3) is verified similarly. The compatibility of (C1) – (C4) with $F_*(S)$ also guarantees that no new identities arise from (B2) – (B5) as consistency conditions. Finally consider the residue equation (C5). It implies

$$2\pi i \operatorname{res} [W_a(\theta + i\pi) W_b(\theta)] = 2\pi i \operatorname{res} E_{ab}^+(\theta + i\pi, \theta) = L_{ab}^+(\theta) - C_{ab} , \quad (\text{B.6})$$

where $L_{ab}^+(\theta)$ is defined as in (2.11). The second residue Eqn. in (2.15) is only indirectly available. In order to be consistent with (C3), the residue $\operatorname{res} C_{ab}(\theta + i\pi, \theta)$ has to be a non-central operator. The resulting expression for $2\pi i \operatorname{res} E_{ab}^-(\theta, \theta + i\pi)$ has the same transformation properties as $L_{ab}^-(\theta) - C_{ab}$, but does not coincide with it. In principle one also has to check the consistency of the relations arising from (B2) – (B4) by taking the residue at $\theta_{12} = \pm i\pi$. The conditions arising are however just those that guarantee the consistency of the algebra $R(S)$ and have been checked before. In summary, this shows that $F(S)$ is a consistently defined associative extension of $F_*(S)$. By means of (B.6) and (3.21) it implements the kinematical residue axiom, and hence may serve as an alternative form factor algebra.

As an aside we remark that the equations (B.2) – (B.5) have an interesting interplay with the anti-homomorphism $\bar{\mathfrak{s}}$: One checks from (2.16) and (B.3), (B.4) that

$$\bar{\mathfrak{s}} W_a(\theta_0) E_{bc}^\pm(\theta_1, \theta_2) = E_{bc}^\pm(\theta_1, \theta_2) \bar{\mathfrak{s}} W_a(\theta_0) , \quad (\text{B.7})$$

consistently for both the expression of the r.h.s of (2.16b). Suppose then that $\bar{\mathfrak{s}}$ has a consistent action on $E_{ab}^\pm(\theta_1, \theta_2)$ and define

$$\tau(E_{ab}^+)(\theta_1, \theta_2) := T^-(\theta_1)_a^m \bar{\mathfrak{s}} E_{nm}^+(\theta_2, \theta_1 - 2\pi i) T^+(\theta_2)_b^n . \quad (\text{B.8})$$

Then $\tau(E_{ab}^+)(\theta_1, \theta_2)$ again satisfies (B.2) – (B.5).

B.2 Reduction of $F(S)$ to $R(S)$

The algebra $F(S)$ can be related to $R(S)$ and hence to the ZF-algebra by a simple reduction prescription. To formulate this reduction note that for $Re\theta_{12} \neq 0$ the (WW) relations and (B.2) imply

$$W_a(\theta_1) W_b(\theta_2) = S_{ab}^{dc}(\theta_{12}) W_c(\theta_2) W_d(\theta_1) \pm 2E_{ab}^{\pm}(\theta_1, \theta_2) , \quad (\text{B.9})$$

for $\pm Re\theta_{12} > 0$. Equivalently, one can define the normal product

$$N[W_a(\theta_1)W_b(\theta_2)] = W_a(\theta_1)W_b(\theta_2) - E_{ab}^{\pm}(\theta_1, \theta_2) , \quad \pm Re\theta_{12} \geq 0 ,$$

and rewrite (B.9) as

$$N[W_a(\theta_1) W_b(\theta_2)] = S_{ab}^{dc}(\theta_{12}) N[W_c(\theta_2) W_d(\theta_1)] . \quad (\text{B.10})$$

By definition $N[W_a(\theta_1)W_b(\theta_2)]$ can be taken to be regular for fixed θ_1 and $0 \leq Im\theta_2 \leq \pi$. Consider now the reduction of $F(S)$ induced by the following prescription

$$C_{ab}(\theta_1, \theta_2) \longrightarrow \pi C_{ab} \delta(\theta_{12} \pm i\pi) , \quad (\text{B.11})$$

where $\delta(\theta)$ is again $\delta(Re\theta) \delta_{Im\theta,0}$. One can easily verify that the r.h.s. is a solution of (C1)–(C3). The pole singularities in (C5) get replaced by the delta distribution singularities at the same positions. Doing this, the extra conditions (C4) can be dropped, provided the relations (S) are changed into (A.30). The relations (B.10) then translate into (A.29), the (TW) relations remain unaffected. Taken together, this means that the form factor algebra $F(S)$ reduced by the prescription (B.11) is isomorphic to the algebra $F_R(S)$ introduced in section A.5. For rapidities that are real modulo $i\pi$, the latter in turn has been seen to be isomorphic to the real rapidity algebra $R(S)$. Finally, the reduction $T^+(\theta)_a^b = -T^-(\theta)_a^b$ leads to an algebra $TZ(S)$ which is a semidirect product of $Z(S)$ and $T(S)$. In summary the relations among the various algebras are

$$F(S) \xrightarrow{\text{red.}} F_R(S) \simeq R(S) \xrightarrow{\text{red.}} TZ(S) \xrightarrow{\text{subalg.}} \supset Z(S) .$$

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References

- [1] M. Karowski and P. Weisz, Nucl. Phys. **B139** (1978) 455.
- [2] F.A. Smirnov, ‘Form Factors in Completely Integrable Models of Quantum Field Theory’, World Scientific, 1992.
- [3] L. Faddeev, E. Sklyanin and L. Takhtajan, Theor. Math. Phys. **40** (1979) 688.
- [4] V. Korepin, N. Bogoliubov and A. Izergin, “QISM and Correlation Functions”, Cambridge University Press, 1993.
- [5] F.A. Smirnov, Int. J. Mod. Phys. **A7** (1992) 813, 839.
- [6] D. Bernard and A. LeClair, Nucl. Phys. **B399** (1993) 709.
- [7] A. LeClair and F.A. Smirnov, Int. J. Mod. Phys. **A7** (1992) 2997.
- [8] F.A. Smirnov, Comm. Math. Phys. **155** (1993) 459.
- [9] F.A. Smirnov, “Form factors, deformed KZ eqn.s and finite gap integration”, preprint Newton Institute, Sept. 92.
- [10] M. Jimbo, T. Kojima, T. Miwa and Y. Quano, J. Phys. **A27** (1994) 3267.
- [11] J. Balog and T. Hauer, Phys. Lett. **337** (1994) 115.
- [12] A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. **393** (1993) 413.
A. Koubek and G. Mussardo, Phys. Lett. **311** (1993) 193.
- [13] E.K. Sklyanin, Nucl. Phys. **326** (1989) 719.
- [14] E.K. Sklyanin, in: “Integrable and Superintegrable Systems”, ed. B. Kupershmidt, World Scientific, Singapore, 1990.
E.K. Sklyanin, J. Phys. **A22** (1989) 3551.

- [15] M. Niedermaier, Nucl. Phys. **424** (1994) 184.
- [16] R. Penrose and W. Rindler, “Spinors and Space-time”, Cambridge University Press, 1984.
- [17] D. Bernard, Comm. Math. Phys. **137** (1991) 181.
D. Bernard and A. LeClair, Comm. Math. Phys. **142** (1991) 99.
- [18] N. Reshetikin and M. Semenov-Tian-Shansky, Lett. Math. Phys. **19** (1990) 133.
- [19] I. Frenkel and N. Reshetikhin, Comm. Math. Phys. **146** (1992) 1.
- [20] N. Reshetikin, Lett. Math. Phys. **26** (1992) 153, 167.
P. Freund and A. Zabrodin, ‘The spectral problem for the q-KZ eqn. and continuous q-Jacobi polynomials’, preprint EFI 93-44, hep-th/9309144.
- [21] B. Feigin, E. Frenkel and N. Reshetikin, “Gaudin model, Bethe Ansatz and correlation functions at the critical level”, hep-th/9402022.
H. Babujian and R. Flume, Mod. Phys. Lett. **A 22** (1994) 2029.
- [22] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, Comm. Math. Phys. **151** (1993) 67.
M. Jimbo and T. Miwa, “Algebraic analysis of solvable lattice models”, preprint RIMS-981, May ’94.
- [23] A. LeClair, Nucl. Phys. **415** (1994) 734.
- [24] S. Lukyanov, ‘Free field representation of massive integrable models’, preprint RU-93-30.
- [25] M. Niedermaier, “On the free field realization of form factors”, preprint MPI-Ph/94-73.
- [26] H. DeVega, in: “Braid group, knot theory and statistical mechanics”, vol.2, ed. C.N. Yang and M.L. Ge, World Scientific, 1994.
- [27] M. Lüscher and U. Wolff, Nucl. Phys. **339** (1990) 222.
- [28] Al. B. Zamolodchikov, Nucl. Phys. **342** (1991) 695.
- [29] P. Dorey, Nucl. Phys. **B358** (1991) 654, Nucl. Phys. **B 374** (1992) 741.
A. Fring and D. Olive, Nucl. Phys. **B379** (1992) 429.
- [30] A. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. **120** (1979) 233.
- [31] R. Haag, “Local Quantum Physics”, Springer TMP, 1992.